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UNIVERSITY OF ALBERTA

POLYNOMIAL INVARIANTS OF FINITE REFLECTION GROUPS

BY

Nicole Marie Anne Lemire

A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of **MASTER OF SCIENCE**.

DEPARTMENT OF MATHEMATICS

Edmonton, Alberta
Spring 1993

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Date: April 7, 1993

Dedicated to:

My parents

ABSTRACT

For an n-dimensional vector space V over a field K of characteristic 0 with symmetric algebra S and for G a finite reflection subgroup of $GL(V)$, Chevalley proved that the algebra of invariants S^G is a graded polynomial algebra over K of transcendental degree n. The degrees of the homogeneous generators for S^G are uniquely determined by G up to order and possess some interesting properties reflecting information about G. The polynomial $\sum_{g \in G} t^{e_1(g)}$ where $e_1(g)$ = the multiplicity of 1 as an eigenvalue of g on V was shown by Solomon to factor linearly into integral factors involving the exponents and hence degrees of G. We examine the factorization of $\sum_{w \in W} \chi(w) t^{e_1(w)}$ where W is a finite real reflection group with character χ. We find that the factorizations of this formula in the case of the determinant character and the trivial character induced from a parabolic subgroup of W consist of linear integral factors involving the exponents of W. We conclude with some examples illustrating conjectures for further results in this area.

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CHAPTER 0

Introduction

In this chapter, we will provide a brief outline of the material to be discussed in this thesis. No attempt will be made to provide precise definitions or statements of results as these will be developed in subsequent chapters.

Let K be a field of characteristic zero and let V be an n-dimensional K-vector space with symmetric algebra $S = S(V)$ and exterior algebra $\Lambda = \Lambda(V)$. A reflection of V is an automorphism of V of finite order which pointwise fixes a hyperplane. Let G be a finite reflection group of V, a finite subgroup of $GL(V)$ generated by reflections. In general, the action of any finite group on V can be uniquely extended to a homogeneous group action on S. In the special case of a finite reflection group G, Chevalley [Ch] proved that the invariants S^G of S under the action of G form a graded polynomial K-algebra generated by n homogeneous algebraically independent invariants over K. It was later shown by Shephard and Todd [ST] that this property characterizes finite reflection groups. That is, if H is a finite subgroup of $GL(V)$ for which S^H is a graded polynomial K-algebra, H is a finite reflection group.

A set of basic invariants for a finite reflection group G is a set of n homogeneous algebraically independent generators for S^G over K. The degrees $d_1 \leq \cdots \leq d_n$ of basic invariants of G are uniquely determined by G and are called the degrees of G. The degrees of a real reflection group W were shown by Coleman to be $m_1 + 1 \leq \cdots \leq m_n + 1$ where the m_i are the exponents of the group. In general the exponents of a finite reflection group G are defined as $m_i := d_i - 1, i = 1, \cdots, n$. The degrees and exponents of G possess some interesting properties. For instance, the product of the degrees is $|G|$ and the sum of the exponents is the number of reflections in G. Solomon [Slm] generalized these two properties to use the

exponents of G to determine information about the multiplicity of 1 as an eigenvalue of group elements. He formed a polynomial $\sum_{k=0}^{n} s_1(k)t^k$ with coefficient $s_1(k)$ as the number of elements of G with 1 as an eigenvalue of multiplicity k and showed that it factored integrally as follows:

$$\sum_{k=0}^{n} s_1(k)t^k = \sum_{g \in G} t^{e_1(g)} = \prod_{i=1}^{n}(t + m_i)$$

where $e_1(g)$ is the multiplicity of 1 as an eigenvalue of g on V. In the case of a real finite reflection group W, Pianzola and Weiss [PW] were able to generalize Solomon's result to primitive m^{th} roots of unity:

$$\frac{1}{|W|}\sum_{k=0}^{n} s_m(k)t^k = \frac{1}{|W|}\sum_{w \in W} t^{e_m(w)} = \prod_{i \in I(m)} \frac{(t + m_i)}{(1 + m_i)}$$

where

$e_m(w)$ = multiplicity of $e^{\frac{2\pi i}{m}}$ as an eigenvalue of w on V

$s_m(k)$ = number of elements of G with $e^{\frac{2\pi i}{m}}$ as an eigenvalue of multiplicity k

$I(m) = \{i | m | m_i + 1\}$

In this thesis, we will examine the following question for χ, a character of a finite real reflection group W: Can $\sum_{w \in W} \chi(w)t^{e_1(w)}$ be factored into a product of integral linear factors involving the exponents of W? Solomon's result shows this to be true for the trivial character. We show that such a result may be obtained for the determinant character and the character of the coset representation of a parabolic subgroup W_J on W. In the particular case of the symmetric group S_{n+1} acting naturally on $\mathbf{R}^{n+1}$, this factorization may be related to the Young diagram corresponding to the parabolic Young subgroup. This allows us to obtain recursively the factorization of $\sum_{w \in W} \chi(w)t^{e_1(w)}$ for at least some of the irreducible characters of S_{n+1}.

In the first seven chapters of this thesis we review the fundamental definitions and results about finite reflection groups and their polynomial invariants. The new

results involving characters will then be proven in Chapter 8. A more detailed outline of the contents of these chapters follows below.

The first chapter will quickly review some basic representation theory which will be needed throughout the thesis. Basic definitions and examples of representations and characters will be given and a few important results will be outlined. The last section contains an overview of the representation theory of the symmetric group which will be required in the last chapter.

Properties of finite reflection groups over a field of characteristic zero are proven in the second chapter. The general definition of a reflection is shown to be equivalent to the more traditional Euclidean reflections in the case of the fields $\mathbf{R}$ and $\mathbf{C}$. Some examples of finite reflection groups are provided. They will be used later to illustrate the results proven in subsequent chapters.

The third chapter deals with the special case of a finite real reflection group W acting effectively on a real Euclidean vector space V. In the first two sections, many productive geometrical connections between the group W and the space V are made. We first illustrate how a finite set of vectors of V called a root system can be used to determine W. The reflections of W are precisely the reflections in the vectors of the root system. The existence of a base for the root system is next proven. The reflections in the vectors of the base are shown to generate W. The hyperplanes of the reflections of W partition V into $|W|$ open regions called chambers. The chambers of W are next shown to be in 1-1 correspondence with the bases of W. Lastly, the action of W on its bases and chambers is proven to be a simple transitive one in which the closure of each chamber is a fundamental chamber for the action of W on V. The third section discusses parabolic subgroups, a special class of subgroups of W which are also finite reflection groups with associated root systems and bases. The last section describes the classification of finite real reflection groups by showing that they are precisely the finite Coxeter groups with positive definite

Coxeter graphs and then outlining the analysis of such Coxeter groups.

Chapter 4 covers graded algebraic structures. In the first section, basic definitions and examples are provided. The second examines Poincaré series of graded K-algebras. The third section identifies S with the polynomial K-algebra $K[X_1, \cdots, X_n]$ and then formally illustrates the unique extension of a group action on V to a homogeneous group action on S and Λ. Then for a finite group H acting on V, S is shown to be a graded S^H-module of rank $|H|$.

The fifth chapter is devoted to the degrees of a finite reflection group G over K. The first section proves Chevalley's Theorem [Ch] showing that S^G is a graded polynomial K-algebra and then uses it to prove that S is a free graded S^G module of rank $|G|$. The second section examines the Poincaré series of S^G. The formulae for the sum and products of the degrees are developed in the third section using the Poincaré series. A criterion for determining a set of basic invariants for G is also used in this section to find the degrees of the finite reflection groups introduced in the second chapter. The fourth section uses a standard Jacobian criterion for algebraic independence and the sum and product formulae to prove Shephard and Todd's converse [ST] to Chevalley's Theorem.

The sixth chapter proves Coleman's theorem showing the connection between the exponents of a finite real reflection group W and the degrees of W. It begins with a discussion of Coxeter elements, elements which are formed by the product of the reflections of the elements of a base for W. Coxeter elements are shown to form a conjugacy class so that they all have the same order h and the same eigenvalues $e^{\frac{2\pi i m_i}{h}}$. The exponents of W are defined as $m_1 \leq \cdots \leq m_n$. We show that a Coxeter element may be embedded into a dihedral subgroup of W so that it acts faithfully as a rotation on a two-dimensional subspace of V. This is used together with geometrical considerations to show that the order of a Coxeter element is $\frac{2N}{n}$ where N is the number of reflections of W and that 1 and h-1 must be exponents of

W. We then proceed to prove Coleman's Theorem. For the special case of a Weyl group, we also show that all the integers between 1 and h-1 relatively prime to h must be exponents. We then use all these results along with the formulae for the sum and product of degrees to determine the exponents and hence degrees of all the finite real irreducible reflection groups.

The seventh chapter is devoted to the proof of Solomon's Theorem [Slm] and the Pianzola-Weiss generalization [PW]. The result hinges on a structure theorem showing that the invariant differential forms $(S \otimes \Lambda)^G$ is a graded exterior algebra generated over S^G by the differentials of a set of basic invariants of G. This is used to compute a formula for the Poincaré series for $(S \otimes \Lambda)^G$. An examination of this formula and another involving the eigenvalues of w leads eventually to the proof of both Solomon's result [Slm] and the Pianzola-Weiss generalization [PW].

Finally the eighth chapter proves the factorization of the formula $\sum_{w \in W} \chi(w) t^{c_1(w)}$ for the determinant character and the coset character $1_{W_J}^W$ of a parabolic subgroup W_J on W. The result is applied to the special case of the symmetric group S_{n+1}. This is then used to find the factorization for certain irreducible characters of S_{n+1}. We conclude with examples illustrating various conjectures for further work in this area.

CHAPTER 1

Basic Representation Theory

Section 1.1: Complete Reducibility of Representations of Finite Groups

This section will review some basic definitions and results in representation theory. We will follow the approach in [Ser]. Let V be an n-dimensional vector space over a field K and let G be a finite group. A **representation** is a group homomorphism $\rho : G \to GL(V)$. A (linear) **group action** is a map $\phi : G \times V \to V$ where $\phi(g,v) \equiv g \cdot v$ satisfying

$$g \cdot (\alpha v + \beta w) = \alpha g \cdot v + \beta g \cdot w$$

$$g \cdot (h \cdot v) = (gh) \cdot v$$

$$1 \cdot v = v$$

where $g,h \in G; v,w \in V; \alpha,\beta \in K$.

There is a one to one correspondence between representations and group actions given by

$$\rho(g)(v) = \phi(g,v).$$

We will make frequent use of this correspondence in what follows.

The space V associated with a representation ρ of G is called the **representation space** of ρ. The **degree** of the representation is the dimension of the space V. Alternatively, V may be regarded as a KG or G-module where the group algebra KG is the K-vector space with basis G in which multiplication is given by:

$$(\sum_{g \in G} a_g g)(\sum_{h \in G} b_h h) = \sum_{k \in G} \sum_{gh=k} a_g b_h k.$$

Here the degree of the representation associated with the group action $v \mapsto g \cdot v$ is the dimension of V over the field K.

Let ρ and γ be two representations of the same group G with representation spaces V and W. Then ρ and γ are said to be **equivalent** if there exists a linear isomorphism $T : V \to W$ which preserves the group action.

$$T \circ \rho(g) = \gamma(g) \circ T, \qquad g \in G$$

The G-modules V and W are said to be isomorphic.

Examples 1.1.1.

(a) Every finite group G affords the 1-dimensional **trivial representation**:

$$\rho(g) = 1 \in \mathbf{C}^* = GL(1, \mathbf{C}), \qquad g \in G$$

(b) Let $G = S_n$ and let G act on the space $\mathbf{R}^n$ with basis $\{e_1, \cdots, e_n\}$ as

$$\sigma \cdot e_i = e_{\sigma(i)}, \qquad \forall\, \sigma \in S_n.$$

This is the **permutation representation** of S_n.

(c) Let G be a finite group $\{g_1, \cdots, g_n\}$. Define an action of G on $K[G]$ by left multiplication, $g \cdot g_i = gg_i$ $\forall g \in G$. This is called the (left) **regular representation** on G.

(d) Let H be a subgroup of the finite group G with cosets $G/H = \{g_1 H, \cdots, g_n H\}$. Then define an action of G on G/H by:

$$g \cdot g_i H = gg_i H \ \forall i = 1, \cdots, n$$

This is called the **coset representation** of H on G. Note that the regular representation of G is the coset representation of $\{1\}$ on G and the trivial representation of G is the coset representation of G on G.

Definition 1.1.2. A **subrepresentation** of a representation $\rho : G \to GL(V)$ is $\rho^W : G \to GL(W)$ where W is a G-stable subspace of V.

$$\rho(g)(w) \in W \qquad \forall g \in G, w \in W.$$

W is a G-**submodule** of V.

Theorem 1.1.3. *(Maschke) Let G be a finite group and V be a n-dimensional K-space with charK $\not|\,|G|$. Let $\rho : G \to GL(V)$ be a representation of G in V and let W be a G-stable subspace of G. Then there exists a G-stable subspace U of V with $V = W \oplus U$.*

Proof: Let W' be a vector space supplement of W in V and let $p : V \to W$ be the corresponding projection. Define

$$p^* = \frac{1}{|G|} \sum_{g \in G} \rho(g) \circ p \circ \rho(g)^{-1}$$

We claim that $p^* : V \to W$ is a G-stable projection of V on W and that $W^* = \ker(p^*)$ is a G-stable supplement.

It is clear that $p^* : V \to W$. Let $w \in W$ then $p\rho(g)^{-1}w = \rho(g)^{-1}(w)$ so that $\rho(g)p(\rho(g))^{-1}w = w$ and hence $p^*w = w$. We find that p^* is a projection of V on W. Since

$$\begin{aligned} \rho(g) \circ p^* \rho(g)^{-1} &= \frac{1}{|G|} \sum_{h \in G} \rho(g)\rho(h)p\rho(h)^{-1}\rho(g)^{-1} \\ &= \frac{1}{|G|} \sum_{h \in G} \rho(gh)p\rho(gh)^{-1} = p^* \end{aligned}$$

we obtain $\rho(g) \circ p^* = p^* \circ \rho(g)$ for all $g \in G$. Let $U = \ker p^*$ and $x \in U$. Then $p^*(\rho(g)x) = \rho(g)p^*x = 0$ so that U is a G-stable supplement of W in V. ∎

Note that for $K = \mathbf{R}$ or $\mathbf{C}$ we can introduce a G-invariant inner product on V by replacing any inner product on V by $(x, y) = \sum_{g \in G} \langle \rho(g)x, \rho(g)y \rangle$. Then the orthogonal complement $W^\perp$ of W with respect to the new inner product is a G-stable supplement of W in V. Note that the invariance of the inner product implies that if $\{\epsilon_i\}_{i=1}^n$ is an orthonormal basis of V, the matrix for $\rho(g)$ with respect to this base is unitary or orthogonal.

Definition 1.1.4. An **irreducible representation** of G on V is a representation $\rho : G \to GL(V)$ such that $V \neq 0$ and there is no proper non-trivial G-invariant

subspace of V. The G-module V is also called irreducible. An irreducible G-module V cannot be written as the direct sum of two G-invariant subspaces.

Theorem 1.1.5. *Every representation is completely reducible. That is, every representation is the direct sum of irreducible representations.*

Proof: Let $\rho : G \to GL(V)$ be a representation of G. Use induction on $\dim V$. Then the result is clear for $\dim V = 0$.

Let $\dim V \geq 1$. If V is irreducible, we are done. Otherwise by 1.1.3, $V = V' \oplus V"$ with $\dim V', \dim V" < \dim V$. By the inductive hypothesis, both V' and $V"$ are the direct sums of irreducible G-modules. ∎

Section 1.2: The character of a representation

Let $\rho : G \to GL(V)$ be a representation of a finite group G in the vector space V over the field $\mathbf{R}$ or $\mathbf{C}$.

Definition 1.2.1. The **character** χ of ρ is the trace of the matrix of $\rho(g)$ with respect to a basis of V.

Proposition 1.2.2. *Let χ be the character of $\rho : G \to GL(V)$. Then*

(a) $\chi(1) = \dim V = n$

(b) χ is invariant under conjugation.

(c) $\chi(g) = \sum_{i=1}^{n} \lambda_i(g)$ where $\lambda_i(g), i = 1 \cdots, n$ are the eigenvalues of g on V.

Proof: (a) is clear.

(b) Let $T(g)$ be the matrix representation of $\rho(g)$. Since $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for any $n \times n$ matrices A and B, we find for $g, h \in G$

$$\chi(ghg^{-1}) = \mathrm{tr}(T(g)^{-1}T(h)T(g)) = \mathrm{tr}(T(g)^{-1}T(g)T(h)) = \mathrm{tr}(T(h)) = \chi(h)$$

(c) Since $\rho(g)$ is diagonalizable as an element of finite order of $GL(V)$, its matrix representation is similar to $\mathrm{diag}(\lambda_1(g), \cdots, \lambda_n(g))$. The result follows from (b). ∎

Examples 1.2.3.

(a) Let $\rho : G \to GL(\mathbf{C}(G))$ be the regular representation of G. Then

$$\chi(g) = \begin{cases} |G| & \text{if } g = 1 \\ 0 & \text{if } g \neq 1 \end{cases}$$

since $gh = h$ for $g, h \in G$ iff $g = 1$.

(b) Let $\varphi : S_n \to GL(\mathbf{R}^n)$ be the permutation representation of S_n on $\mathbf{R}^n$. Then its character is

$$\chi(\sigma) = \#\text{ of fixed points of } \sigma$$

$$= \#\text{ of 1-cycles of } \sigma$$

since $\sigma \cdot \epsilon_i = \epsilon_{\sigma(i)} = \epsilon_i$ iff $\sigma(i) = i$.

(c) Let $\rho_H : G \to GL(V)$ be the coset representation of $H \leq G$. We claim that its character is $\chi_H(g) = \frac{1}{|H|} \sum_{x \in G} 1_H(xgx^{-1})$ where

$$1_H(y) = \begin{cases} 1 & \text{if } y \in H \\ 0 & \text{if } y \notin H. \end{cases}$$

Let $G/H = \{g_1 H, \cdots, g_n H\}$. Indeed

$$\chi_H(g) = \#\text{ of cosets } g_i H \text{such that } gg_i H = g_i H$$

$$= \#\text{ of cosets } g_i H \text{ such that } g_i^{-1} g g_i \in H$$

But for all $h \in H, g_i^{-1} g g_i \in H$ iff $(g_i h)^{-1} g (g_i h) \in H$. Since $G = \cup_{i=1}^n g_i H$, the result follows.

We will now state a few theorems about characters and representations without proof. The proofs may be found in [Ser]. They will provide background for the work that follows.

An inner product can be defined on the space $R(G)$ of all class functions of G where

$$R(G) = \{f : G \to \mathbf{C} | f(xgx^{-1}) = f(g) \forall x, g \in G\}$$

via $(\varphi, \psi) = \frac{1}{|G|} \sum_{g \in G} \varphi(g) \overline{\psi}(g)$.

Proposition 1.2.4. *Two representations of a group with the same character are equivalent.*

Theorem 1.2.5. *The irreducible characters form an orthonormal basis for $R(G)$. Hence the number of irreducible representations of G is the number of conjugacy classes of G.*

Theorem 1.2.6. *Let $\rho : G \to GL(V)$ be a representation of G in V with character χ. Let $\chi_1, \cdots, \chi_n$ be the characters corresponding to the irreducible representations $\rho_1, \cdots, \rho_n$ with spaces $W_1, \cdots, W_n$. Then*

$$V \simeq m_1 W_1 \oplus \cdots \oplus m_n W_n$$

$$\chi \simeq m_1 \chi_1 + \cdots + m_n \chi_n$$

where $m_i = (\chi, \chi_i)$. V is irreducible iff $(\chi, \chi) = 1$.

Section 1.3: Representation Theory of the Symmetric Group

We will be particularly interested in the representations of the symmetric group S_n. The cycle type of $\sigma \in S_n$ may be given by a partition $\lambda = (\lambda_1, \cdots, \lambda_k)$ of n in which the λ_i are strictly decreasing and sum to n. If $\sigma \in S_n$ has cycle type λ, then it may be written as a product of k disjoint cycles in which the ith cycle has length i. Each conjugacy class of S_n is given by the permutations of S_n of a particular cycle type. Hence the number of conjugacy classes of S_n is the number of partitions of n. By Theorem 1.2.5, this is the number of irreducible characters of S_n.

To examine the irreducible characters of S_n more closely, we require some definitions and notation.

Definitions 1.3.1. Let $\lambda = (\lambda_1, \cdots, \lambda_k) \mapsto n$ and $\mu = (\mu_1, \cdots, \mu_k) \mapsto n$ be partitions of n.

(a) The **Young diagram of shape** λ is an array of n boxes arranged in k left justified rows in which row i has λ_i boxes for $1 \leq i \leq k$.

(b) A **standard Young tableau of shape** λ is a Young diagram in which each of the numbers from 1 through n appears in exactly one of the boxes of the diagram so that the entries in the rows and columns are strictly increasing.

(c) A **semi-standard Young tableau of shape** λ **and content** μ is a Young diagram in which μ_i copies of i for $1 \leq i \leq m$ are distributed among the boxes of the diagram so that the entries of the rows are weakly increasing and the entries of the columns are strictly decreasing.

(d) The **Young subgroup** S_λ associated with λ is $S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_k}$.

Example 1.3.2. Let $\lambda = (2,2,1)$. The Young subgroup is $S_2 \times S_2 \times S_1 \cong S_2 \times S_2$. As we see in Figure 1.3.1, there are 5 standard Young tableaux and 2 semistandard tableaux of shape $(3,2)$ and content λ.

Let the coset representation of S_λ on S_n have module M_λ and permutation character 1_{S_λ} and let the irreducible representation associated with λ have module V_λ and character χ^λ. Although the coset representations of S_λ on S_n provide the correct number of representations, they are not all irreducible. However they are easier to work with and information about the irreducible characters can be obtained from the permutation characters.

Theorem 1.3.3. *The dimension of* V_λ *is the number of standard Young tableaux of shape* λ.

Proof: [Sgn]

Definition 1.3.4. The dominance ordering on partitions of n.
Let $(\lambda_1, \cdots, \lambda_k) \mapsto n$ and $(\mu_1, \cdots, \mu_m) \mapsto n$. Then λ dominates μ written $\lambda \trianglerighteq \mu$ iff

$$\sum_{j=1}^{i} \lambda_j \geq \sum_{j=1}^{i} \mu_j, \text{for all } i \geq 1$$

If $i > k (i > m)$ then take $\lambda_i = 0 (\mu_i = 0)$.

Theorem 1.3.5. *The multiplicity of V_λ in M_μ is given by $K_{\lambda\mu} = \#$ of semistandard Young tableaux of shape λ and content μ. That is $M_\mu \cong \oplus_{\lambda \trianglerighteq \mu} K_{\lambda\mu} V_\lambda$ or*

$$1_{S_\mu} = \sum_{\lambda \trianglerighteq \mu} K_{\lambda\mu} \chi^\lambda.$$

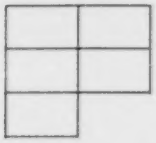
Proof: [Sgn]

Example 1.3.6. For $\mu = (2,2,1) \mapsto 5$ The partitions $\lambda \trianglerighteq \mu$ are:

$$\lambda_1 = (2,2,1), \lambda_2 = (3,1,1), \lambda_3 = (3,2), \lambda_4 = (4,1), \lambda_5 = (5)$$

so that $1_{S_{(2,2,1)}} = \chi^{(2,2,1)} + 2\chi^{(3,1,1)} + 2\chi^{(3,2)} + 2\chi^{(4,1)} + \chi^{(5)}$.

FIGURE 1.3.1: EXAMPLES OF YOUNG DIAGRAMS AND YOUNG TABLEAUX



Young diagram of shape (2,2,1)

1	2
3	4
5	

1	3
2	4
5	

1	2
3	5
4	

1	3
2	5
4	

1	4
2	5
3	

Standard Young Tableaux of shape (2,2,1)

1	1	2
2	3	

1	1	3
2	2	

Semi-standard Young Tableaux of shape (3,2) and content (2,2,1)

CHAPTER 2

Finite Reflection Groups

In this section we will define reflections and finite reflection groups, prove a few important properties and give some examples. We will assume that V is an n-dimensional vector space over a field K of characteristic zero.

Definition 2.1.1. A **reflection** is a non-trivial automorphism of V which has finite order and pointwise fixes a hyperplane (subspace of dimension n-1).

Let s be a reflection of order m. Since $s^m = 1$, the minimal polynomial of s has distinct roots in the algebraic closure of K so that s is diagonalizable. We know that 1 is an eigenvalue of muliplicity n-1 because s pointwise fixes a hyperplane. The remaining eigenvalue λ is distinct from 0 and 1 since s is a non-trivial automorphism. In fact $\lambda = \det(s) \in K$. Hence we may decompose V as $E_\lambda \oplus E_1$ where E_λ, E_1 are respectively the λ-eigenspace and 1-eigenspace of s. It is clear that λ is a primitive mth root of unity. Let H be the hyperplane E_1. We find that s acts on $V = K\alpha \oplus H$ via

$$s\alpha = \lambda\alpha \qquad s|_H = 1_H$$

where λ is a primitive mth root of unity. We call s a reflection in $\alpha \in V$.

Definition 2.1.2. A **finite reflection group** G is a finite subgroup of $GL(V)$ which is generated by reflections.

Proposition 2.1.3. *The reflections of G which fix a hyperplane W together of V together with the identity form a cyclic subgroup of G.*

Proof: The reflections of G which fix W and 1 form a subgroup

$$G_W = \{g \in G | g|_W = 1_W\}.$$

By Maschke's theorem, there exists a G_W-stable supplement $K\alpha$ of W in V.

Assume without loss of generality that K is algebraically closed. Let $x \in G_W$. Then α is an eigenvector of x with eigenvalue $\lambda(x)$. We claim that G_W is isomorphic to a subgroup of K^* under the map $x \mapsto \lambda(x)$.

Let $x, y \in G_W$. Then λ is a homomorphism since

$$\lambda(xy)\alpha = \lambda(x)\lambda(y)\alpha.$$

λ is injective since $x \in \ker\lambda$ implies that $x(\alpha) = \alpha$ so that $x = 1$. Hence G_W is isomorphic to a subgroup of the multiplicative group K^* and hence is cyclic. ∎

Corollary 2.1.4. *Let $0 \neq \alpha \in V$. The reflections in α of G and 1 form a cyclic subgroup of G.*

Proof: $G_\alpha = \{g \in G | g(\alpha) \in K\alpha\}$ is a subgroup of G. By Maschke's theorem there exists a G_α-stable supplement W of K_α in V. By definition, the elements of G_α which pointwise fix the hyperplane W are precisely the reflections of α in G and the identity. Hence $G_\alpha \cap G_W$ is the required cyclic subgroup. ∎

In the case of $K = \mathbf{R}$ or $\mathbf{C}$, V may be endowed with a symmetric (hermitian) G-invariant bilinear form $(\cdot,\cdot) : V \times V \to K$. Then G may be regarded as a subgroup of the orthogonal or unitary group of V. A G-invariant subspace W has as G-invariant supplement the orthogonal complement of $W^\perp$ with respect to this inner product. Hence the hyperplane of a reflection in α is $H_\alpha = \{x \in V | (x, \alpha) = 0\}$.

For a real reflection group G, reflections must be of order 2 with -1 as the distinct eigenvalue. Here a reflection in α takes the form

$$s_\alpha x = x - 2\frac{(x,\alpha)}{(\alpha,\alpha)}\alpha.$$

Note that s_α is indeed an orthogonal transformation with $(s_\alpha x, s_\alpha y) = (x, y)$ for all $x, y \in V$.

Examples 2.1.5. [H2], [Spr]

$K = \mathbf{R}$.

(a) Let $V = \mathbf{R}^2$. The dihedral group of order 2m consisting of the orthogonal transformations which preserve a regular m-gon centred at the origin. $\mathcal{D}_2^m$ contains m rotations through multiples of $\frac{2\pi}{m}$ and m reflections about the diagonals of the polygon. It is generated by the reflections s_α, s_β where $\alpha = (\sin\frac{\pi}{m}, -\cos\frac{\pi}{m})$ and $\beta = (0,1)$. Note that $s_\alpha s_\beta$ is a rotation through $\frac{2\pi}{m}$.

(b) Let $\epsilon_1, \cdots, \epsilon_n$ be the standard basis of $\mathbf{R}^n$. The symmetric group S_n can be regarded as a subgroup of the group $O(n, \mathbf{R})$. $\sigma \in S_n$ acts on $\mathbf{R}^n$ via $\epsilon_i \mapsto \epsilon_{\sigma(i)}$. Note that (i, j) acts as the reflection $s_{\epsilon_i - \epsilon_j}$. Since S_n is generated by the transpositions $(i, i+1)$, it is a finite reflection group.

(c) $\mathcal{A}_{n-1}, n \geq 2$. Let $V = (\sum_{i=1}^n \epsilon_i)^\perp$. Let S_n act on V as in (b). Note that a basis for V is $\{\epsilon_i - \epsilon_{i+1}\}$ so that V is stable under the action of S_n.

(d) $\mathcal{B}_n, n \geq 2$. Let $V = \mathbf{R}^n$. Let S_n act on V as in (2). The other generating reflections are given by :

$$\epsilon_i \mapsto -\epsilon_i \qquad \epsilon_j \mapsto \epsilon_j, j \neq i$$

The group generated by these sign changes is isomorphic to $(\mathbf{Z}_2)^n$. Since it intersects S_n trivially and is normalized by S_n, the reflection group $\mathcal{B}_n$ generated by S_n and these sign-changes is isomorphic to the semi-direct product of S_n by $(\mathbf{Z}_2)^n$ Thus $\mathcal{B}^n$ is of order $2^n n!$.

(e) $\mathcal{D}_n, n \geq 4$ Another reflection group acting on $\mathbf{R}^n$ is found as a subgroup of index 2 in the group $\mathcal{B}_n$. S_n normalizes the subgroup consisting of even numbers of sign changes generated by the reflections $s_{\epsilon_i + \epsilon_j}, i \neq j$. This subgroup is isomorphic to $(\mathbf{Z}_2)^{n-1}$. The semi-direct product of S_n with this subgroup is also a reflection group of order $2^{n-1} n!$ denoted by $\mathcal{D}_n$.

(f) $G(n, m, d)$. Let $K = \mathbf{C}$, $V = \mathbf{C}^n$ Let $n, m, d \geq 1$ with $d|m$. The group $G(n, m, d)$ is generated by all transformations of the type $\epsilon_i \mapsto \theta_i \epsilon_{\sigma(i)}$ where

$\theta_i^m = 1, \sigma \in S_n$ and $(\theta_1 \cdots \theta_n)^d = 1$. The reflections of the group are of the form:

$$\epsilon_i \mapsto \zeta \epsilon_i \qquad \epsilon_j \mapsto \epsilon_j, j \neq i$$

or

$$\epsilon_i \mapsto \zeta \epsilon_j \qquad \epsilon_j \mapsto \zeta^{-1} \epsilon_i \qquad \epsilon_k \mapsto \epsilon_k, k \neq i, j$$

where $\zeta^m = 1$. This is a group of order $dm^{n-1}n!$. This example generalizes the previous examples. Note that $S_n \cong G(n,1,1), \mathcal{B}_n \cong G(n,2,2), \mathcal{D}_n \cong G(n,2,1)$, and $\mathcal{D}_2^m \cong G(2,m,1)$.

CHAPTER 3

Finite Real Reflection Groups

In this chapter, we present the special properties of finite real reflection groups. In the first section, we discuss the bijective correspondence between finite effective real reflection groups and root systems of V. We also show the existence of a base Δ for a root system Φ. The second section examines the bijection between the bases of a root system of W and the chambers of W. The action of W on the bases and chambers is shown to be a simple transitive one for which the closure of each chamber is a fundamental chamber of the action of W on V. A special class of subgroups of W called parabolic subgroups is studied in the third section. Parabolic subgroups are the subgroups of W which are also finite real reflection groups with associated root systems and bases. They form a lattice of 2^n subgroups. The last section shows that finite reflection groups are finite Coxeter groups with associated positive definite Coxeter graphs. This information is used to classify all finite real reflection groups. This chapter follows the treatment of this material in [H1], [H2], [GB] and [Car].

Section 3.1: Root Systems

Let W be a finite real reflection group acting on the real n-dimensional vector space V. By introducing a W-invariant inner product on V, we may assume that it is a subgroup of the orthogonal group $O(V)$. By Maschke's theorem we may write V as $V = V^W \oplus U$ where V^W is the space of W-invariants of V and $U = (V^W)^\perp$.

Definition 3.1.1. A subgroup W of $O(V)$ with $V^W = 0$ is called **effective**.

W can be regarded as a subgroup of $O(U)$ under the restriction homomorphism $w \mapsto w\,|_U$ since w acts as the identity on V^W. Hence without loss of generality we may assume that W is effective.

Example 3.1.2. In 2.1.5, S_n regarded as a subgroup of $O(\mathbf{R}^n)$ is not effective since $V^{S_n} = \text{span}\{\sum_{i=1}^{n} \epsilon_i\}$. However it may be identified with $\mathcal{A}_{n-1}$ which acts on $(V^{S_n})^{\perp}$ and is effective.

We may associate each reflection of W with a vector of V.

Definition 3.1.3. A **root** of a reflection w is a non-zero vector α with $w = s_\alpha$.

Note that the choice of a root corresponding to a reflection is not unique since $s_{c\alpha} = s_\alpha$ for all $c \neq 0$.

Definition 3.1.4. A **root system** of V is a subset Φ of V which satisfies the following axioms.

R S1 Φ is finite, $V = \text{span}\Phi$, $0 \notin \Phi$.

R S2 $\Phi \cap \mathbf{R}\alpha = \{\pm\alpha\}$

R S3 $s_\alpha \Phi = \Phi$ for all $\alpha \in \Phi$.

A root system is called **crystallographic** if it also satisfies the following axiom.

R S4 $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbf{Z}$ for $\alpha, \beta \in \Phi$.

Theorem 3.1.5. *Root systems of V correspond bijectively with effective finite reflection subgroups W of $O(V)$. The finite effective reflection group W determines a root system $\Phi(W)$ as the set of roots of unit length of conjugates of the generating reflections of W. The root system Φ determines a finite effective reflection subgroup of $O(V)$,*

$$W(\Phi) = \langle s_\alpha | \alpha \in \Phi \rangle.$$

Before proving this theorem, the following lemmas are required.

Lemma 3.1.6. *Let $w, s_\alpha \in W \leq O(V)$. Then $s_{w\alpha} = w s_\alpha w^{-1} \in W$ so that $w\alpha$ is a root.*

Proof: Let $y \in V$ and $x = w^{-1}y$. Then

$$w s_\alpha w^{-1}(y) = w s_\alpha x = wx - \frac{2(x,\alpha)}{(\alpha,\alpha)} w\alpha$$

$$= wx - 2\frac{(wx, w\alpha)}{(w\alpha, w\alpha)}w\alpha = s_{w\alpha}(y) \quad \blacksquare$$

Lemma 3.1.7. *$W = \langle s_{\alpha_i} | i = 1, \cdots, k \rangle$ is effective if and only if $V = \text{span}\{\alpha_i | i = 1, \cdots, k\}$.*

Proof: We show that $V^W = \cap_{i=1}^k H_{\alpha_i}$ where $H_{\alpha_i} = (\mathbf{R}\alpha_i)^\perp$. This follows from

$$x \in \cap_{i=1}^k H_{\alpha_i} \Leftrightarrow s_{\alpha_i} x = x \text{ for all } i \Leftrightarrow wx = x \text{ for all } w \in W$$

since W is generated by s_{α_i}.

W is effective if and only if $V^W = 0$. From results in linear algebra, we obtain

$$\sum_{i=1}^k \mathbf{R}\alpha_i = \sum_{i=1}^k (H_{\alpha_i})^\perp = (\cap_{i=1}^k H_{\alpha_i})^\perp = 0^\perp = V \quad \blacksquare$$

Proof of the theorem: Let W is a finite effective reflection subgroup of $O(V)$. $\Phi(W)$ is finite since W has a finite number of generating reflections and W is a finite group. It spans V by Lemma 3.1.7 and clearly $0 \notin \Phi(W)$. Thus RS1 is satisfied. RS2 is satisfied by definition. RS3 follows from Lemma 3.1.6 and the fact that elements of W preserve lengths.

Let Φ be a root system for V. Then $W(\Phi) = \langle s_\alpha | \alpha \in \Phi \rangle$ is an effective reflection subgroup of $O(V)$ by Lemma 3.1.7. Since span$\Phi = V$, $w\alpha = \alpha$ for all $\alpha \in \Phi$ implies that w acts as the identity on V. Hence $W(\Phi)$ acts faithfully on the finite set Φ and may be identified with a subgroup of S_Φ. This implies that W is finite. $\blacksquare$

We will now assume that W is a finite reflection group determined by the root system Φ. Since span$\Phi = V$, we wish to choose a convenient basis $\Delta \subset \Phi$ of V.

Definition 3.1.8. A subset Δ of Φ is called a **base** if

(B1) Δ is a basis of V.

(B2) Each root β can be written as $\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha$ with coefficients all non-negative or all non-positive.

The roots of Δ are called simple. By (B1), $|\Delta| = n$ and the expression for β in (B2) is unique. If all the coefficients k_α for β are positive (negative), β is called respectively positive (negative). Denote the collections of positive and negative roots relative to Δ by Φ^+ and Φ^-. Note that $\Phi^- = -\Phi^+$. Unfortunately there is one problem with the definition of base: that of existence. Our goal is to prove the following theorem.

Theorem 3.1.9. *Φ has a base.*

The proof will be constructive in nature. For each vector $\gamma \in V$ define

$$\Phi^+(\gamma) = \{\alpha \in \Phi | (\gamma, \alpha) > 0\}.$$

Geometrically, this is the set of vectors lying on the positive side of the hyperplane orthogonal to γ. By Baire's category theorem, $\cup_{\alpha \in \Phi} H_\alpha \neq V$. Call $\gamma \in V$ **regular** if $\gamma \in V - \cup_{\alpha \in \Phi} H_\alpha$; that is , $(\gamma, \alpha) \neq 0 \ \forall \alpha \in \Phi$. When γ is regular, $\Phi = \Phi^+(\gamma) \cup \Phi^-(\gamma)$.

Take γ to be a regular element of V and let $\Delta(\gamma)$ be a minimal subset of $\Phi^+(\gamma)$ such that each element of $\Phi^+(\gamma)$ may be expressed as a non-negative linear combination of elements of $\Delta(\gamma)$. We will show that $\Delta(\gamma)$ forms a base for Φ. We first require two lemmas. Note that $x \in \Phi^+(\gamma)$ iff $(x, \gamma) > 0$ and $x \in \Phi^-(\gamma)$ iff $(x, \gamma) < 0$.

Lemma 3.1.10. *If $\alpha, \beta \in \Delta(\gamma)$ with $\alpha \neq \beta$ and $k_\alpha, k_\beta > 0$ then $x = k_\alpha \alpha - k_\beta \beta$ is neither positive nor negative.*

Proof: If x were positive then $x = k_\alpha \alpha - k_\beta \beta = \sum_{\delta \in \Delta} m_\delta \delta$ with all $m_\delta \geq 0$. If $k_\alpha \leq m_\alpha$ then

$$0 = (m_\alpha - k_\alpha)\alpha + (m_\beta + k_\beta)\beta + \sum_{\delta \neq \alpha, \beta} m_\delta \delta$$

so that

$$0 = (\gamma, (m_\alpha - k_\alpha)\alpha + (m_\beta + k_\beta)\beta + \sum_{\delta \neq \alpha,\beta} m_\delta \delta)$$

$$\geq k_\beta(\gamma, \beta) > 0$$

which is a contradiction. If $k_\alpha > m_\alpha$ then

$$(k_\alpha - m_\alpha)\alpha = (m_\beta + k_\beta)\beta + \sum_{\delta \neq \alpha,\beta} m_\delta \delta.$$

Dividing through by $k_\alpha - m_\alpha$, we obtain α as a non-negative linear combination of $\Delta(\gamma) - \{\alpha\}$ which contradicts the minimality of $\Delta(\gamma)$. Hence x is not positive.

If x is negative, $-x$ would be positive so that a repetition of the above argument with α and β interchanged would produce a contradiction. ∎

Lemma 3.1.11. *Let $\alpha, \beta \in \Delta(\gamma)$ with $\alpha \neq \beta$. Then $s_\alpha(\beta) \in \Phi^+(\gamma)$ and $(\alpha, \beta) \leq 0$.*

Proof: Since $s_\alpha(\beta) = \beta - 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha \in \Phi$ and the coefficient of α is positive, we find by the previous lemma that $(\beta, \alpha) \leq 0$ and $s_\alpha(\beta) \in \Phi^+(\gamma)$. ∎

Note that geometrically $(\alpha, \beta) \leq 0$ means that the angle θ between α and β is obtuse since $\cos\theta = \frac{(\alpha,\beta)}{|\alpha||\beta|}$.

Proof of the theorem: We prove that $\Delta(\gamma)$ forms a base for Φ for some regular element γ of V and that all bases for Φ have this form.

We first show that $\Delta(\gamma)$ is a linearly independent set. Suppose not and rearrange the dependence relation so that it takes the form

$$\sum k_\alpha \alpha = \sum m_\beta \beta$$

where $k_\alpha > 0, m_\alpha > 0$ Then setting $\epsilon = \sum k_\alpha \alpha$, we obtain

$$O \leq (\epsilon, \epsilon) \leq \sum_{\alpha,\beta} k_\alpha m_\beta (\alpha, \beta) \leq 0$$

so that $\epsilon = 0$. But then $0 = (\epsilon, \gamma) = k_\alpha(\alpha, \gamma)$ so that all $k_\alpha = 0$. Similarly $m_\beta = 0$. In fact this argument shows that a set of vectors which lie strictly on one side of a hyperplane and form pairwise obtuse angles must be linearly independent.

Since each element of $\Phi^+(\gamma)$ ($\Phi^-(\gamma)$) is a non-negative (non-positive) linear combination of elements of $\Delta(\gamma)$ and Φ spans V, we find that $\Delta(\gamma)$ spans V and satisfies (B2). We have shown that $\Delta(\gamma)$ forms a base for Φ.

Conversely, let Δ be a base for Φ. We claim there exists $\gamma \in V$ such that $(\gamma, \alpha) > 0$ for all $\alpha \in \Delta$. Indeed take $\gamma = \sum_{\beta \in \Delta^*} \beta$ where $\Delta^* = \{\beta_i\}$ is the dual basis for $\Delta = \{\alpha_i\}$ with $(\alpha_i, \beta_j) = \delta_{ij}$. Then by (B2), γ is regular, $\Phi^+ \subset \Phi^+(\gamma)$ and $\Phi^- \subset \Phi^-(\gamma)$ so that equality must hold in each case. Since Δ is a basis, Δ is a minimal subset of $\Phi^+ = \Phi^+(\gamma)$ with the property that each element of $\Phi^+(\gamma)$ can be written as a non-negative linear combination of the elements of Δ. We claim that $\Delta = \Delta(\gamma)$. We will actually show that the change of basis matrix A from the basis $\Delta(\gamma)$ to the basis Δ is a permutation matrix.

Note that A has non-negative entries as each root of Δ is a non-negative linear combination of roots of $\Delta(\gamma)$. Similarly $B = A^{-1}$ has non-negative entries. Let $a_1, \cdots, a_n$ be the rows of A and $b_1, \cdots, b_n$ be the columns of B. Then $AB = I$ implies $a_1 \cdot b_i = 0$ for all $j = 2, \cdots, n$. If a_1 has a non-zero entry in the jth column , the jth row entry of $b_2, \cdots, b_n$ must be 0. Since $b_2, \cdots, b_n$ are linearly independent, a_1 can have at most 1 non-zero entry. Similarly each row of A can have at most 1 non-zero entry. Since A is non-singular, A has exactly one positive entry in each column and in each row. Each root of Δ is a positive multiple of a root of $\Delta(\gamma)$. Hence by RS2, A is a permutation matrix so that $\Delta = \Delta(\gamma)$. ∎

Example 3.1.12. [GB] Let $W = \mathcal{D}_2^m$ be the dihedral group of order 2m. The root system is

$$\Phi = \{(\cos\frac{k\pi}{m}, \sin\frac{k\pi}{m}) | k = 0, \ldots, 2m-1\}$$

Choosing $t = (\sin\frac{\pi}{4m}, \cos\frac{\pi}{4m})$ we find

$$\Phi^+ = \{(\cos\frac{k\pi}{m}, \sin\frac{k\pi}{m}) | 0 \leq k \leq m-1\}$$

$$\Delta = \{(1,0), (\cos\frac{(m-1)\pi}{m}, \sin\frac{(m-1)\pi}{m})\}$$

Section 3.2: Bases and Chambers

The hyperplanes $H_\alpha, \alpha \in \Phi$ partition V into finitely many regions. The connected components of $V - \cup_\alpha H_\alpha$ are called the **chambers** of V. Each regular $\gamma \in V$ belongs to precisely one chamber, denoted by $C(\gamma)$. $C(\gamma) = C(\gamma')$ if and only if γ and γ' lie on the same side of each hyperplane $H_\alpha, \alpha \in \Phi$ if and only if $\Phi^+(\gamma) = \Phi^+(\gamma')$ or $\Delta(\gamma) = \Delta(\gamma')$. Hence the chambers are in one to one correspondence with the bases. The **fundamental chamber relative to** Δ is $C(\Delta) = C(\gamma)$ if $\Delta = \Delta(\gamma)$. Then $C(\Delta) = \cap_{\alpha \in \Delta} H_\alpha^+$ where $H_\alpha^+ = \{\gamma \in V | (\gamma, \alpha) > 0\}$.

In fact the action of W on the chambers and the bases of Φ is compatible with this correspondence. Since $wH_\alpha^+ = H_\alpha^+$, W acts on the chambers of Φ via $wC(\gamma) = C(w\gamma)$ where $w \in W$ and γ is regular. Since $w\Phi^+(\gamma) = \Phi^+(w\gamma)$, W permutes bases via $w\Delta(\gamma) = \Delta(w\gamma)$.

We now prove a few lemmas about simple roots. Fix a base Δ of Φ.

Lemma 3.2.1. *Let α be a simple root. Then the reflection s_α permutes the positive roots other than α.*

Proof: Let $\beta \in \Phi^+, \beta \neq \alpha$. If $\beta \in \Delta$ then $\beta \in \Phi^+$ by Lemma 3.1.10.

If $\beta \notin \Delta$, then $\beta = \sum_{\gamma \in \Delta} k_\gamma \gamma$ with at least two $k_\gamma > 0$. Assume $k_\delta > 0$ with $\delta \neq \alpha$. Then

$$\begin{aligned} s_\alpha(\beta) &= \sum_{\gamma \in \Delta} k_\gamma s_\alpha(\gamma) \\ &= k_\delta \delta + \sum_{\gamma \neq \delta} k_\gamma \gamma - \sum_{\gamma \in \Delta} \frac{2(\gamma, \alpha)}{(\alpha, \alpha)} \alpha \end{aligned}$$

Since $s_\alpha(\beta)$ has at least one positive coefficient , k_δ, it is positive. Clearly $s_\alpha(\beta) \neq \alpha$ since $s_\alpha(-\alpha) = \alpha$. ∎

Corollary 3.2.2. *Set $\delta = \frac{1}{2} \sum_{\beta \in \Phi^+} \beta$. Then $s_\alpha(\delta) = \delta - \alpha \quad \forall \alpha \in \Delta$.*

Lemma 3.2.3. *Let $\alpha_1, \cdots, \alpha_t \in \Delta$ (not necessarily distinct). Write $s_i = s_{\alpha_i}$. If $s_1 \cdots s_{t-1}(\alpha_t) \in \phi^-$ then for some index $1 \leq r < t$,*

(i) $s_{r+1} \cdots s_{t-1}(\alpha_t) = \alpha_r$

(ii) $s_{r+1} \cdots s_t = s_r \cdots s_{t-1}$

(iii) $s_1 \cdots s_t = s_1 \cdots s_{r-1} s_{r+1} \cdots s_{t-1}$.

Proof: Let $\beta_i = s_{i+1} \cdots s_{t-1}(\alpha_t), 0 \leq i \leq t-2; \beta_{t-1} = \alpha_t$. Since $\beta_0 \in \Phi^-$ and $\beta_{t-1} \in \Phi^+$, there exists a minimal index r such that $\beta_r \in \Phi^+$. But then $s_r(\beta_r) = \beta_{r-1} \in \Phi^-$ so that Lemma 3.2.1 implies that $\beta_r = \alpha_r$ and $s_{r+1} \cdots s_{t-1}(\alpha_t) = \alpha_r$ giving (i). By Lemma 3.1.6, we obtain $s_r = (s_{r+1} \cdots s_{t-1}) s_t (s_{t-1} \cdots s_{r+1})$ so that (ii) and (iii) follow as required. ∎

Corollary 3.2.4. *If $s = s_1 \cdots s_t$ is an expression for $w \in W$ in terms of simple reflections with t minimal then $s(\alpha_t) \in \Phi^-$.*

Proof: The result follows from 3.2.3(iii). ∎

We may now prove that W permutes the bases (or chambers) of Φ in a simple transitive fashion and that W is generated by the simple reflections relative to Δ.

Theorem 3.2.5. *Let Δ be a base for Φ.*

(a) W acts transitively on chambers of Φ. That is, for γ a regular element of V, there exists $w \in W$ with $(w(\gamma), \alpha) > 0$ for all $\alpha \in \Delta$.

(b) W permutes the bases of Φ. That is, if Δ' is another base for Φ, there exists $w \in W$ with $w(\Delta') = \Delta$.

(c) Every root $\alpha \in \Phi$ is W-conjugate to an element of Δ. That is , there exists $w \in W$ with $w(\alpha) \in \Delta$.

(d) $W = \langle s_\alpha | \alpha \in \Delta \rangle$.

(e) W acts simply transitively on bases. That is, if $w(\Delta) = \Delta, w \in W$ then $w = 1$.

Proof: Let $W' = \langle s_\alpha | \alpha \in \Delta \rangle$. We will prove (a) to (c) for W' and then show $W' = W$.

(a) We use the element $\delta = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$ and claim that an element $w \in W'$ for which $(w(\gamma), \delta)$ is maximal satisfies the hypothesis. As W' is finite, such an

element must exist.

If $\alpha \in \Delta$, then $s_\alpha w \in W'$. Using Corollary 3.2.2, we obtain

$$(w(\gamma),\delta) \geq (s_\alpha w(\gamma),\delta) = (w(\gamma),\delta-\alpha) = (w(\gamma),\delta) - (w(\gamma),\alpha)$$

so that $(w(\gamma),\alpha) \geq 0$. But if $(w(\gamma),\alpha) = 0$ then $(\gamma, w^{-1}(\alpha)) = 0$ which contradicts the fact that γ is regular. Hence there exists $w(\gamma) \in \cap_{\alpha\in\Delta} H_\alpha = C(\Delta)$ so that $w(C(\gamma)) = C(\Delta)$.

(b) Since W permutes the chambers of Φ by (a), it also permutes the bases of Φ.

(c) Let $\Delta = \Delta(\gamma)$ for some regular γ. If $\alpha \in \Delta$, we let $w = 1$. If $\alpha \notin \Delta$ then $\Delta \cup \{\alpha\}$ is linearly dependent so from the proof of 3.1.9, we find that there exists $\alpha_1 \in \Delta$ with $(\alpha,\alpha_1) > 0$. We construct a sequence of elements $\{a_i\}$ of $\Phi^+(\gamma)$ with $(a_{i+1},\gamma) < (a_i,\gamma)$.

Set $a_1 = s_1(\alpha) = \alpha - \frac{2(\alpha,\alpha_1)}{(\alpha_1,\alpha_1)}\alpha_1$. By 3.1.10, $a_1 \in \Phi^+$ and

$$(a_1,\gamma) = (\alpha,\gamma) - \frac{2(\alpha,\alpha_1)}{(\alpha_1,\alpha_1)}(\alpha_1,\gamma) < (\alpha,\gamma).$$

If $a_1 \in \Delta$, set $s = s_1 \in W'$. Otherwise apply the same process to a_1 to obtain $\alpha_2 \in \Delta$ and

$$a_2 = s_2 a_1 = s_2 s_1 \alpha \in \Phi^+(\gamma) \qquad (a_2,\gamma) < (a_1,\gamma).$$

If $a_2 \in \Delta$, set $w = s_2 s_1$, otherwise continue this process. Since $\Phi^+(\gamma)$ is finite, there exists $a_r = s_r \cdots s_1 \alpha \in \Delta$. Set $w = s_r \cdots s_1 \in W'$.

(d) To prove $W' = W$, it suffices to show that every reflection $s_\alpha \in W'$ for $\alpha \in \Phi$. By (c), there exists $w \in W'$ with $\beta = w(\alpha) \in \Delta$. Then $s_\beta = w s_\alpha w^{-1}$ so that $s_\alpha = w^{-1} s_\beta w \in W'$.

(e) Let $w(\Delta) = \Delta$ with $w \neq 1$. By (d) we may write w as a product of a minimal number of simple reflections. But then $w(\alpha) \in \Phi^+$ for all $\alpha \in \Delta$. This contradicts Corollary 3.2.4. ∎

In summary, W acts simply transitively on bases and chambers by the maps :

$$\Delta(\gamma) \mapsto \Delta(w\gamma) \qquad C(\gamma) \mapsto C(w\gamma)$$

and a complete list of the bases (chambers) of Φ is given by $\{\Delta(w\gamma)|w \in W\}$ and $\{C(w\gamma)|w \in W\}$ so that the bases (chambers) are in one to one correspondence with the elements of the group W.

Since W is generated by simple reflections, we may write $w \in W$ as $w = s_{\alpha_1} \cdots s_{\alpha_t}$ where $\alpha_i \in \Delta$ and t is minimal. This expression for w is called **reduced** and the **length** of w relative to Φ is defined to be $l(w) = t$. By definition, $l(1) = 0$. We will characterize the length of $w \in W$ in a different way. Define

$$n(w) = |\Phi^+ \cap w^{-1}(\Phi^-)|$$

$$= \text{ number of positive roots } \alpha \text{ with } w(\alpha) \in \Phi^-.$$

We will use the following lemma to show that $l(w) = n(w)$.

Lemma 3.2.6. *Let $\alpha \in \Delta$ and $w \in W$*

(i) $n(s_\alpha w) = n(w) + 1$ *if* $w^{-1}(\alpha) \in \Phi^+$

(ii) $n(s_\alpha w) = n(w) - 1$ *if* $w^{-1}(\alpha) \in \Phi^-$

(iii) $n(ws_\alpha) = n(w) + 1$ *if* $w(\alpha) \in \Phi^+$.

(iv) $n(ws_\alpha) = n(w) - 1$ *if* $w(\alpha) \in \Phi^-$

Proof: By 3.2.1, s_α changes the sign of only two roots α and $-\alpha$. Then $n(s_\alpha w) = n(w) \pm 1$ and $n(ws_\alpha) = n(w) \pm 1$. But $n(s_\alpha w) = n(w) + 1$ if and only if $\alpha \in w(\Phi^+)$ proving (i) and (ii) and $n(ws_\alpha) = n(w) + 1$ if and only if $w(\alpha) \in \Phi^+$ which proves (iii) and (iv). ∎

Theorem 3.2.7. *For all $w \in W, l(w) = n(w)$.*

Proof: We use induction on $l(w)$. If $l(w) = 0$ we have $w = 1$ so that $n(w) = 0$. Assume the result holds for $\tau \in W$ with $l(\tau) < l(w)$. Write $w = s_{\alpha_1} \cdots s_{\alpha_t}$ in

reduced form and set $\alpha = \alpha_i$. By Corollary 3.2.4, $w(\alpha) \in \Phi^-$ so that by Lemma 3.2.6, $n(ws_\alpha) = n(w) - 1$. But $l(ws_\alpha) = l(w) - 1 < l(w)$. We may apply the inductive hypothesis to ws_α to obtain $l(w) = n(w)$. ∎

We now look more closely at the simple transitive action of W on the chambers of Φ. Note that the fundamental chamber $C(\Delta)$ has closure

$$\overline{C}(\Delta) = \cap_{\alpha \in \Delta}(H_\alpha \cup H_\alpha^+).$$

We show that $\overline{C}(\Delta)$ is a fundamental domain for the action of W on V, that is $|W \cdot x \cap \overline{C}(\Delta)| = 1$.

Theorem 3.2.8.

(a) If $w\lambda = \mu$ for $\lambda, \mu \in \overline{C}(\Delta), w \in W$ then $\lambda = \mu$ and w is a product of simple reflections fixing λ. In particular, if $\lambda \in C(\Delta)$, $w\lambda = \lambda$ implies that $w = 1$.

(b) $\overline{C}(\Delta)$ is a fundamental domain for the action of W on V.

Proof:

(a) The result is proved by induction on $l(w) = n(w)$. If $n(w) = 0$, $w = 1$ and the result is trivial. In the case $n(w) > 0$, there exists $\alpha \in \Delta$ with $w(\alpha) \in \Phi^-$. ws_α satisfies the inductive hypothesis since $n(ws_\alpha) = n(w) - 1$ by Lemma 3.2.6. But $\lambda, \mu \in \overline{C}(\Delta)$ implies

$$0 \geq (\mu, w\alpha) = (w\lambda, w\alpha) = (\lambda, \alpha) \geq 0$$

so that $(\lambda, \alpha) = 0$ and $s_\alpha \lambda = \lambda$. Then $ws_\alpha \lambda = \mu$ and by the inductive hypothesis $\lambda = \mu$ and ws_α is a product of simple reflections fixing λ; hence so is w. In the case $w\lambda = \lambda$ where $\lambda \in C(\Delta)$, if a simple reflection s_α fixes λ we obtain $(\lambda, \alpha) = 0$. The contradiction implies $w = 1$.

(b) Each $v \in V$ lies in the closure of some chamber $C(w^{-1}\Delta)$. Hence, by the definition of the action of W on the chambers, $wv \in \overline{C}(\Delta)$. The uniqueness of the W-conjugate of v in $\overline{C}(\Delta)$ follows from (a). ∎

Section 3.3: Parabolic Subgroups of W

We now examine a special class of subgroups of the reflection group associated to the root system Φ which are also finite reflection groups. Fix a base Δ for Φ. Let J be a subset of Δ, $V_J = \text{span}(J)$, $\Phi_J = \Phi \cap V_J$ and $W_J = \langle s_\alpha | \alpha \in J \rangle$. The following proposition finds a natural root system and base for W_J.

Proposition 3.3.1. *Φ_J is a root system in V_J. J is a base for Φ_J. The reflection group associated with Φ_J is W_J.*

Proof: RS1 and RS2 are clearly satisfied by Φ_J. If $\alpha, \beta \in \Phi_J$ then

$$s_\alpha(\beta) = \beta - 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha \in \Phi \cap V_J$$

so that Φ_J satisfies RS2 and hence is a root system.

J is clearly a basis for V_J. Since $J \subset \Delta$, each root can be expressed uniquely as a linear combination of roots in J with all coefficients non-negative or non-positive. The reflection group of Φ_J is $\langle s_\alpha | \alpha \in J \rangle = W_J$. ∎

The subgroups W_J are called **parabolic subgroups**. Note that replacing the base Δ with its conjugate $w\Delta$ replaces W_J with its conjugate $W_{wJ} = wW_Jw^{-1}$.

Example 3.1.2. For the root system $A_n = \{\epsilon_i - \epsilon_j | i \neq j, 1 \leq i, j \leq n+1\}$ with base $\{\alpha_i = \epsilon_i - \epsilon_{i+1} | i = 1, \ldots, n\}$. We have seen that the corresponding reflection group is S_{n+1}. In this case, the parabolic subgroups are isomorphic to the Young subgroups $S_{\lambda_1} \times \cdots \times S_{\lambda_k}$. If $J = \{\alpha_{i_1} \cdots, \alpha_{i_1+\lambda_1-2}, \cdots, \alpha_{i_k} \cdots, \alpha_{i_k+\lambda_k-2}\}$ where $i_j + \lambda_j - 1 < i_{j+1}$ for $j = 1, \cdots, k-1$, then $W_J \cong S_{\lambda_1} \times \cdots \times S_{\lambda_k}$.

Parabolic subgroups are useful for proving results about finite reflection groups. The following theorem characterizes the fixed point subspace V^w of an element w of the group.

Theorem 3.3.3. *Let $w \in W$ and $U = V^w = \{w | wv = v\}$. Then w is a product of simple reflections corresponding to roots in $U^\perp$.*

Proof: Let $v_1, \cdots, v_k$ be a basis for U. We proceed by induction on k to show that w is a product of simple reflection corresponding to roots orthogonal to $v_1, v_2, \cdots, v_k$.

The result is true for k=1 by Theorem 3.2.8(a). Choose Δ such that $v_k \in \overline{C}(\Delta)$ and let $J = \{\alpha \in \Delta | (\alpha, v_k) = 0\}$. Consider $V = V_J \oplus V_J^{\perp}$ and let $v_i = v_i' + v_i''$ where $v_i' \in V_J, v_i'' \in V_J^{\perp}$. Since $wv_k = v_k$, Theorem 3.2.8(a) says that $w = s_{\alpha_1} \cdots s_{\alpha_m}$ where $(\alpha_i, v_k) = 0$ for $i = 1, \cdots, k$. Hence $w \in W_J$ so that $w \mid_{V_J^{\perp}} = Id_{V_J^{\perp}}$ which implies that $w(v_i'') = v_i''$. Since also $w(v_i) = v_i$ we obtain $w(v_i') = v_i'$. But W_J is a reflection group acting on V_J and the element $w \in W_J$ fixes $v_1', \cdots, v_{k-1}'$. So by induction , w is a product of reflections corresponding to roots orthogonal to $v_1', \cdots, v_{k-1}'$. Since these roots are orthogonal to $v_1, \cdots, v_k$, the result follows. ∎

This theorem helps to clear up an ambiguity in the definition of the root system Φ specified by a finite effective reflection group. It turns out that the reflections of W are of the form s_α for some $\alpha \in \Phi$.

Proposition 3.3.4. *$\{s_\alpha | \alpha \in \Phi^+\}$ is the set of reflections of W.*

Proof: Let s be a reflection of W with reflecting hyperplane H pointwise fixed by s. By the theorem, s is a product of reflections of the form s_α where $\alpha \in H^{\perp}$. But s_α fixes H pointwise iff $H = H_\alpha$ so $s = s_\alpha$. Since $s_\alpha = s_{-\alpha}$ we have the result. ∎

The following theorem presents a picture of the collection of parabolic subgroups of W.

Theorem 3.3.5. *The subgroups W_J form a lattice of 2^n subgroups of W.*

Proof: We show that $J \mapsto W_J$ defines a lattice isomorphism between the subsets of a base Δ and the parabolic subgroups of W. This map is clearly surjective.

Suppose $W_J = W_K$ but J and K are distinct subsets of Δ. Assume there exists $\alpha \in K - J$. Then for all $x = x' + x" \in V$ where $x' \in V_J, x" \in V_J^{\perp}$,

$$-2\frac{(x, \alpha)}{(\alpha, \alpha)}\alpha = s_\alpha(x) - x = s_\alpha(x') - x' \in V_J$$

Take x with $(x,\alpha) \neq 0$ so that $\alpha \in V_J$. But this contradicts the fact that Δ is linearly independent. Then $J = K$.

Clearly the subgroup generated by W_J and W_K is $W_{J\cup K}$. It remains to show that $W_J \cap W_K = W_{J\cap K}$. We have $W_{J\cap K} \subset W_J \cap W_K$ by definition.

Let $w \in W_J \cap W_K$. Then $w(v) = v$ for all $v \in V_J^\perp$ and for all $v \in V_K^\perp$. So w fixes $V_J^\perp + V_K^\perp$. We claim that $V_J^\perp + V_K^\perp = V_{J\cap K}^\perp$. Clearly $V_J^\perp + V_K^\perp \subset V_{J\cap K}^\perp$. Also

$$\begin{aligned}
\dim(V_J^\perp + V_K^\perp) &= \dim V_J^\perp + \dim V_K^\perp - \dim(V_J^\perp \cap V_K^\perp) \\
&= \dim V_J^\perp + \dim V_K^\perp - \dim(V_J \cap V_K)^\perp \\
&= n - \dim(V_J \cap V_K) \\
&= n - \dim V_{J\cap K} \\
&= \dim V_{J\cap K}^\perp
\end{aligned}$$

so that $V_J^\perp + V_K^\perp = V_{J\cap K}^\perp$. By the theorem since w fixes every vector of $V_{J\cap K}^\perp$, it is a product of reflections corresponding to roots in $V_{J\cap K}$. Thus $w \in W_{J\cap K}$. ∎

Section 3.4: Classification of Finite Real Reflection Groups

It turns out that the finite real reflection groups are in one to one correspondence with finite Coxeter groups. Recall the following definition.

Definition 3.4.1. A finite Coxeter group G is a finite group with presentation

$$G = \langle g_i \in G | (g_i g_j)^{m_{ij}} = 1 \qquad m_{ii} = 1 \qquad m_{ij} \in \mathbf{N} \rangle$$

We first prove that a finite reflection group W is a finite Coxeter group.

Theorem 3.4.2. $W = \langle s_\alpha | (s_\alpha s_\beta)^{m_{\alpha\beta}} = 1 \quad m_{\alpha\alpha} = 1 \quad m_{\alpha\beta} \in \mathbf{N};\ \alpha,\beta \in \Delta \rangle$

Proof: We prove the result by contradiction. Let $s_1 \cdots s_k = 1$ where $s_i = s_{\alpha_i}$, $\alpha_i \in \Delta$ be a relation of minimal length k which cannot be obtained from the given relations. We claim that k is even. Indeed, dets_i = -1 so that $\det(s_1 \cdots s_k) = (-1)^k$ implies that k = 2m. We may write

$$s_1 \cdots s_{m+1} = s_{2m} \cdots s_{m+2}.$$

Since $l(s_1 \cdots s_{m+1}) < m+1$, there exists an integer $j \leq m$ with $s_1 s_2 \cdots s_j(\alpha_{j+1}) \in \Phi^-$ by Lemma 3.2.6 and induction. Lemma 3.2.3 uses the above relations to show that there exists $1 \leq i < j$ with

$$s_{i+1} \cdots s_j(\alpha_{j+1}) = \alpha_i \qquad s_{i+1} \cdots s_{j+1} = s_i \cdots s_j$$

The latter is equivalent to a relation of length 2(j-i+1). If this length is less than 2m the relation follows from the given relations. Using this and the relations $s_i^2 = 1$ we obtain

$$s_1 \cdots s_k = s_1 \cdots s_{i-1} s_{i+1} \cdots s_j s_{j+2} \cdots s_k$$

The relation $s_1 \cdots s_{i-1} s_{i+1} \cdots s_j s_{j+2} \cdots s_k = 1$ has length less than k so that it may be deduced from the given relations. Hence $s_1 \cdots s_k = 1$ can also be deduced from the given relations. By contradiction 2(j-i+1) = 2m so that j=m, i=1. Hence $s_2 \cdots s_m(\alpha_{m+1}) = \alpha_1$.

However $s_1 \cdots s_k = 1$ is equivalent to $s_2 \cdots s_k s_1 = 1$. By a repetition of the above process we find

$$s_3 \cdots s_{m+2} = s_2 \cdots s_{m+1}$$

But if this relation could be deduced from the given relations we could use it to show that

$$s_1 \cdots s_k = s_1 s_3 \cdots s_{m+1} s_{m+3} \cdots s_k$$

and then $s_1 s_3 \cdots s_{m+1} s_{m+3} \cdots s_k = 1$ could be deduced from the given relations. By contradiction, we obtain another relation of length 2m which cannot be deduced from the given relations and may be written in the form

$$s_3(s_2 \cdots s_{m+1}) s_{m+2} s_{m+1} \cdots s_4 = 1$$

Again as before $s_2 \cdots s_m(\alpha_{m+1}) = \alpha_1$ so that $\alpha_1 = \alpha_3$.

We may now write $s_1 \cdots s_k = 1$ in equivalent forms by cyclically permuting $\alpha_1, \cdots, \alpha_k$ to deduce that

$$\alpha_1 = \alpha_3 = \cdots = \alpha_{2m-1} = \alpha$$

$$\alpha_2 = \alpha_4 = \cdots = \alpha_{2m} = \beta$$

Then we have $(s_\alpha s_\beta)^m = 1$ and $m = qm_{\alpha\beta}$ for some q. But then $(s_\alpha s_\beta)^m = 1$ may be deduced from $(s_\alpha s_\beta)^{m_{\alpha\beta}} = 1$. By contradiction, the theorem is proved. ∎

A Coxeter group may be associated with a diagram which completely determines the group.

Definition 3.4.3. A **Coxeter graph** Γ is a finite (undirected) graph whose vertices are identified with the generators of the Coxeter group. The edge joining α and β is labelled with the order $m_{\alpha\beta}$ of $s_\alpha s_\beta$. Conventions:

(1) $m_{\alpha\alpha} = 1$ is not labelled.

(2) If $m_{\alpha\beta} = 2$, the vertices α and β are not joined by an edge.

(3) If $m_{\alpha\beta} = 3$, the vertices α and β are joined by an unlabelled edge.

The following proposition clarifies the notion that a Coxeter graph completely determines a finite reflection group.

Proposition 3.4.4. *Let W_1, W_2 be finite reflection groups acting on the euclidean spaces V_1 and V_2 with the same Coxeter graph. Then there is an isometry of V_1 onto V_2 which induces an isomorphism of W_1 and W_2.*

Proof: Fix bases Δ_1 and Δ_2 for W_1 and W_2. Assume without loss of generality that Δ_1 and Δ_2 consist of unit vectors. Then the statement that W_1 and W_2 have the same Coxeter graph implies that Δ_1 and Δ_2 may be written as $\Delta_1 = \{\alpha_1, \cdots, \alpha_n\}$ and $\Delta_2 = \{\beta_1, \cdots, \beta_n\}$ so that $m(\alpha_i, \alpha_j) = m(\beta_i, \beta_j)$ which means that

$$(\alpha_i, \alpha_j) = \cos\theta = -\cos(\pi/m(\alpha_i, \alpha_j)) = -\cos(\pi/m(\beta_i, \beta_j)) = (\beta_i, \beta_j)$$

since Δ_1 and Δ_2 are bases for V_1 and V_2. Thus we may extend $\gamma : \alpha_i \to \beta_i$ to an isometry. Then the map $\tilde{\gamma} : s_{\alpha_i} \to s_{\beta_i}$ may be extended to an isomorphism of W_1 and W_2 since $\tilde{\gamma}(s_{\alpha_i}) = s_{\gamma\alpha_i} = \gamma s_{\alpha_i} \gamma^{-1}$. ∎

We now reduce our study of finite reflection groups to a smaller class of groups.

Definition 3.4.5. A finite reflection group W (or its associated root system Φ) is called **irreducible** if its Coxeter graph is connected.

Let $\Gamma_1, \cdots, \Gamma_k$ be the connected components of Γ and let Δ_i, S_i be the corresponding sets of simple roots and simple reflections. If $\alpha \in \Delta_i$ and $\beta \in \Delta_j (i \neq j)$, note that $m(\alpha, \beta) = 2$ so that s_α and s_β commute.

Proposition 3.4.6. *Let W be a finite reflection group acting on V have Coxeter graph Γ with connected components $\Gamma_1, \cdots, \Gamma_r$ and corresponding subsets $\Delta_1, \cdots, \Delta_r$ of its base Δ. Then*

(i) $W = \prod_{i=1}^{r} W_{\Delta_i}$ where each W_{Δ_i} is a finite irreducible real reflection group.

(ii) $V = \oplus_{i=0}^{r} V_i$ where $V_0 = V^W, V_i = V_{\Delta_i}, i = 1, \cdots, r$.

(iii) Let $w_i \in W_{\Delta_i}$. Then $w_i \mid_{V_j} = Id_{V_j}$ for all $j \neq i$.

(iv) Each V_i is W-stable.

Proof:

(i) Use induction on r. The case r=1 is clear. Since the simple reflections in S_i commute with those in S_j when $i \neq j$, we see that W_{Δ_i} and W_{Δ_j} centralize each other and hence each is normal in W. Also $W = W_{\Delta_1} \cdots W_{\Delta_r}$ since the simple reflections generate W. By the inductive hypothesis, $W_{\Delta - \Delta_i}$ is the direct product $\prod_{j \neq i} W_{\Delta_j}$. But by 3.3.5, $W_{\Delta_i} \cap W_{\Delta - \Delta_i} = W_\phi = 1$. Hence the product is direct.

(ii) By Maschke's theorem $V = V^W \oplus (V^W)^\perp$ where $(V^W)^\perp$ is the orthogonal complement of V^W with respect to the W-invariant inner product on V. We find that $(V^W)^\perp = V_\Delta = \oplus_{i=1}^{r} V_{\Delta_i}$. Hence $V = \oplus_{i=0}^{r} V_i$ is an orthogonal direct sum with $V_0 = V^W$ and $V_i = V_{\Delta_i}$ for $i = 1, \cdots, r$.

(iii) Since $w_i \in W_{\Delta_i}$ implies that $w_i \mid_{V_{\Delta_i}^{\perp}} = Id_{V_{\Delta_i}^{\perp}}$ and $V_{\Delta_i}^{\perp} = \oplus_{j \neq i} V_i$ we find that $w_i \mid_{V_j} = Id_{V_j}$ for all $j \neq i$.

(iv) Let $w = w_1 \cdots w_k$ where $w_i \in W_{\Delta_i}$ and let $v_j \in V_j$. Then $wv_j = w_j v_j \in V_j$ so that V_j is W-stable. ∎

We now associate a symmetric n × n matrix A to the Coxeter graph Γ with n vertices $\{\alpha_1, \cdots, \alpha_n\}$. Define $a_{ij} = -\cos \frac{\pi}{m(\alpha_i, \alpha_j)}$. The Coxeter graph is called positive definite iff A is positive definite. When Γ is the Coxeter graph of a finite reflection group W with base $\Delta = \{\alpha_i\}_{i=1}^{n}$ where the roots are normalized to have unit length, the matrix A is positive definite since then $a_{ij} = (\alpha_i, \alpha_j)$. Hence the approach in the classification of all irreducible finite reflection groups is to list all positive definite connected Coxeter graphs and to show that each is associated with a finite reflection group. We will not provide the details of this process. The result is described in the Theorem below.

Theorem 3.4.7. *The only positive definite connected Coxeter graphs are* A_n, B_n, D_n, F_4, G_2, H_3, H_4, D_2^m *as shown in* **Figure 3.4.1.** *Each of these graphs gives rise to an irreducible root system with a corresponding finite reflection group as shown in* **Table 3.4.1.**

Finite reflection groups corresponding to crystallographic root systems are called Weyl groups and are important in Lie theory. The effect of the added condition RS4 on the root system Φ is to force $s_\alpha(\beta)$ to be an integral combination of α and β. The roots in Φ are then all $\mathbf{Z}$ - linear combinations of Δ and hence form a lattice. Hence the associated Weyl group is called crystallographic since it stabilizes the lattice Φ in V.

We prove a necessary condition for a finite reflection group to be crystallographic.

Proposition 3.4.8. *If W is crystallographic then $m_{\alpha,\beta} = 2,3,4,$ or 6 if $\alpha \neq \beta$ in Δ.*

Proof: If $\alpha \neq \beta$, s_α and s_β generate a dihedral subgroup of order $2m_{\alpha\beta}$ which acts on the plane $\mathbf{R}\alpha + \mathbf{R}\beta$ by a rotation through $2\pi/m(\alpha,\beta)$ while fixing

$$(\mathbf{R}\alpha)^\perp \cap (\mathbf{R}\beta)^\perp = (\mathbf{R}\alpha + \mathbf{R}\beta)^\perp$$

pointwise. Representing $s_\alpha s_\beta$ as a matrix with respect to an appropriate basis for V we find that its trace is (n-2) + 2cosθ. Since the trace of $(s_\alpha s_\beta)$ must be an integer, we find that the only possibilities for cosθ are $-1, -\frac{1}{2}, 0, \frac{1}{2}$ which correspond to $m(\alpha,\beta) = 2,3,4,6$. ∎

The following theorem summarizes all of the above results.

Theorem 3.4.9. *Let W be a finite reflection group with base Δ and Coxeter graph Γ acting on the n-dimensional real Euclidean space V. Let the connected components of the Coxeter graph $\Gamma_i, i = 1, \cdots, k$ correspond to the subsets $\Delta_i, i = 1, \cdots, k$ of Δ. Then $W = \prod_{i=1}^{k} W_{\Delta_i}$ and $V = \oplus_{i=0}^{r} V_i$ with $V_0 = V^W, V_i = V_{\Delta_i}, i = 1, \cdots, r$ such that:*

(a) Each W_{Δ_i} is one of the groups $\mathcal{A}_n$, $n \geq 1$; $\mathcal{B}_n$, $n \geq 2$; $\mathcal{D}_n$, $n \geq 4$, $\mathcal{F}_4$, $\mathcal{D}_2^m$, $m \geq 5$, $m \neq 6$; $\mathcal{G}_2$; $\mathcal{H}_3$, $\mathcal{H}_4$; $\mathcal{E}_6$, $\mathcal{E}_7$, $\mathcal{E}_8$.

(b) W is crystallographic iff $\mathcal{D}_2^m$, $\mathcal{H}_3$ and $\mathcal{H}_4$ do not appear among the direct factors.

(c) W acts effectively on V iff $V^W = 0$.

(d) Each V_i is W-stable.

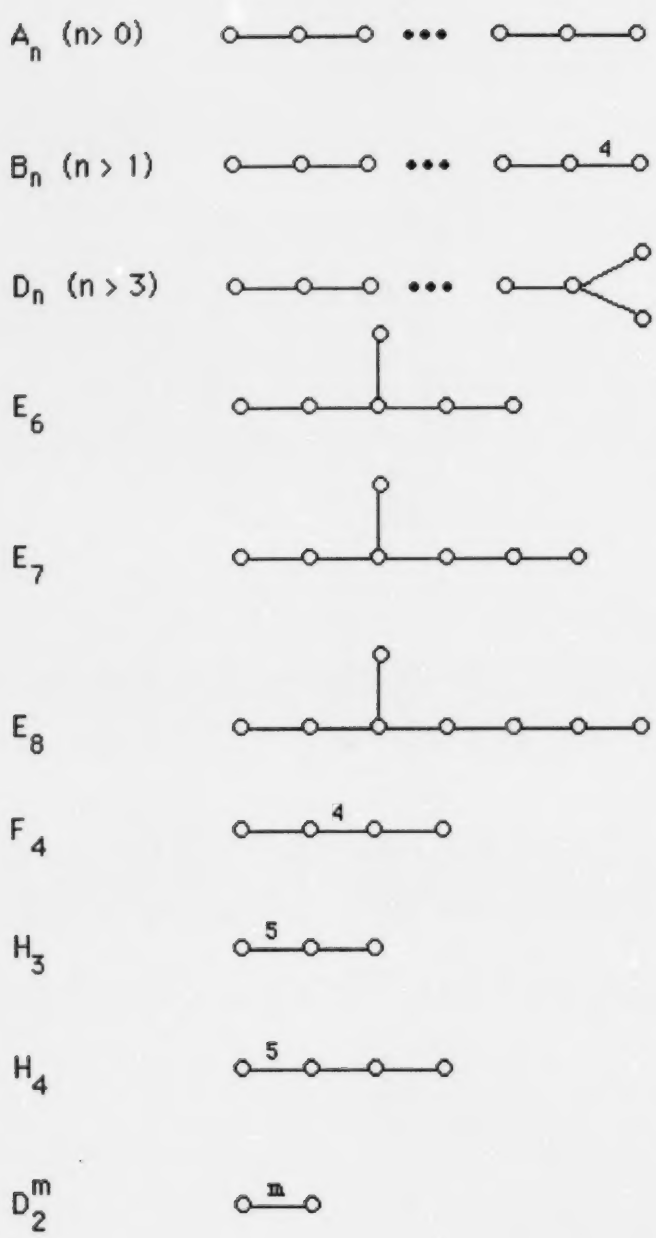


FIGURE 3.4.1: Positive Definite Irreducible Coxeter Graphs

Table 3.4.1: Root Systems

Graph	Group	$\lvert\Phi\rvert$	Root System Φ
A_n	$\mathcal{A}_n$	n^2+n	$\pm(\epsilon_i-\epsilon_j), 1\le j<i\le n+1$
B_n	$\mathcal{B}_n$	$2n^2$	$\pm\epsilon_i, 1\le i\le n; \pm\epsilon_i\pm\epsilon_j, 1\le j<i\le n.$
D_n	$\mathcal{D}_n$	$2n(n-1)$	$\pm\epsilon_i\pm\epsilon_j, 1\le j<i\le n.$
D_2^m	$\mathcal{D}_2^m$	$2m$	$(\cos j\pi/m, \sin j\pi/m), 0\le j\le 2m-1.$
G_2	$\mathcal{G}_2$	12	$\pm(\epsilon_i-\epsilon_j), 1\le j<i\le 3; \pm(1,-2,1),$ $\pm(-2,1,1), \pm(1,1,-2).$
H_3	$\mathcal{H}_3$	30	$\pm\epsilon_i, 1\le i\le 3;$ $\beta(\pm(2\alpha+1), \pm1, \pm2\alpha)$, and all even permutations of coordinates.
H_4	$\mathcal{H}_4$	120	$\pm\epsilon_i, 1\le i\le 4; (1/2)(\pm1,\pm1,\pm1,\pm1);$ $\beta(\pm2\alpha, 0, \pm(2\alpha+1), \pm1)$, and all even permutations of coordinates.
F_4	$\mathcal{F}_4$	48	$\pm\epsilon_i, 1\le i\le 4;$ $\pm\epsilon_i\pm\epsilon_j, 1\le j<i\le 4;$ $(1/2)\sum_1^4 a_i\epsilon_i, a_i=\pm1$
E_8	$\mathcal{E}_8$	240	$\pm\epsilon_i\pm\epsilon_j, 1\le j<i\le 8; (1/2)\sum_1^8 a_i\epsilon_i,$ $a_i=\pm1, \prod_1^8 a_i=-1.$
E_7	$\mathcal{E}_7$	126	Those roots of $\mathcal{E}_8$ orthogonal to $(1/2)(1,1,1,1,1,1,1,-1).$
E_6	$\mathcal{E}_6$	72	Those roots of $\mathcal{E}_7$ orthogonal to $\epsilon_8-\epsilon_7$

CHAPTER 4

Graded Algebraic Structures

In this chapter we examine the concept of graded algebraic structures. Later this will allow us to characterize the symmetric algebra S and its subalgebra of invariants S^G. The exposition is based on that in [B1], [B2], and [B3].

Section 4.1: Definitions and Examples

Definition 4.1.1. A collection of subgroups of a commutative group G, $(G_\lambda)_{\lambda\in\mathbf{Z}}$ is called a **Z -grading** of G if G is the direct sum of the collection. If such a Z-grading exists, G is called a graded group. An element x of G is homogeneous of degree $\lambda\in\mathbf{Z}$ if and only if $x\in G_\lambda$.

From now on we will omit the reference to $\mathbf{Z}$ in graded algebraic structures. Note that every element of a graded group G can be written uniquely as a sum of its homogeneous components.

To extend the concept of a grading to the category of rings, modules and algebras, we must ensure that the grading on the additive group is compatible with the algebraic structure of the category.

Definition 4.1.2.

(i) A **graded ring** is a ring R with a grading $(R_\lambda)_{\lambda \in \mathbf{Z}}$ on its additive group such that $R_\mu R_\lambda \subset R_{\mu+\lambda}$ for all $\mu, \lambda \in \mathbf{Z}$.

(ii) A **graded (left) R-module** is a (left) R-module M with a grading $(M_\mu)_{\mu \in \mathbf{Z}}$ on its additive group such that $R_\mu M_\lambda \subset M_{\mu+\lambda}$ for all $\mu, \lambda \in \mathbf{Z}$.

(iii) A **graded R-algebra** is an algebra A with a grading $(A_\mu)_{\mu \in \mathbf{Z}}$ on its additive group so that it is both a graded ring and a graded R-module.

The definition of a graded ring R forces R_0 to be a subring. We already know that R_0 is a commutative subgroup of R with $R_0 R_0 \subset R_0$. Then to see that R_0 is a subring of R it suffices to show that $1 \in R_0$. If $1 = \sum_{\lambda \in \mathbf{Z}} e_\lambda$ where $e_\lambda \in \mathbf{Z}$ is the decomposition of 1 into its homogeneous components then for $x \in R_\mu$, $x = \sum_{\lambda \in \mathbf{Z}} x e_\lambda$. But then $\mu = \deg(x e_\lambda) = \mu + \lambda$ implies that $x = x e_o$. In particular $1 = e_o \in R_0$. Note that a graded R-module can then be also considered to be a graded R_0-module.

An easy example of a graded ring is $A = R[X_1, ..., X_n]$, the polynomial ring in n commuting indeterminates over the commutative ring R. A has a grading $(A_\lambda)_{\lambda \in \mathbf{Z}}$ where

$$A_\lambda = \{r X_1^{k_1} \cdots X_n^{k_n} | \sum_{i=1}^{n} k_i = \lambda, k_i \geq 0, r \in R\}$$

Note that $A_0 = R$ is indeed a subring of A. In fact A can also be considered as an example of a graded R-algebra where the grading on R is the trivial grading. (i.e.

$R_0 = R, R_\lambda = 0$ for $\lambda \neq 0$.)

Another example of a graded algebra is the **tensor algebra** $T(V)$ of an n-dimensional vector space V over a field K. The tensor algebra is defined as

$$T(V) = \sum_{n \geq 0} T^n(V)$$

where $T^n(V)$ is the n- fold tensor product of V and $T^0(V) = K$. If K is given the trivial grading, we see that $T(V)$ is a graded K-module with componentwise scalar multiplication and addition. As multiplication in $T(V)$ is defined on the generators of $T^p(V)$ and $T^q(V)$ for $p, q \geq 0$ by

$$(x_1 \otimes \cdots \otimes x_p)(x_{p+1} \otimes \cdots \otimes x_{p+q}) = x_1 \otimes \cdots \otimes x_{p+q} \in T^{p+q}(V)$$

and

$$(\alpha)(v_p) = (v_p)(\alpha) = \alpha v_p \in T^p(V) \forall\ \alpha \in T^0(V), v_p \in T^p(V)$$

we see that $T(V)$ is also a graded ring. Hence $T(V)$ is a graded K-algebra with grading $(T^n(V))_{n \geq 0}$.

We need conditions under which subrings, submodules and subalgebras of graded rings, modules and algebras are also graded structures. These will follow from the following proposition.

Proposition 4.1.3. *Let R be a graded ring and M a graded R-module with grading $(M_\lambda)_{\lambda \in \mathbf{Z}}$. Let N be an R-submodule of M. The following are equivalent:*

(a) $N = \oplus_{\lambda \in \mathbf{Z}} N \cap M_\lambda$.

(b) *The homogeneous components of an element of N belong to N.*

(c) *N is generated by homogeneous elements.*

Proof: (a) and (b) are equivalent since every element of N can be written uniquely as a sum of homogeneous elements. By the same reasoning (a) implies (c).

Let us show that (c) implies (b). Suppose $(z_i)_{i\in I}$ is a set of homogeneous generators of N with $\deg(z_i)= \delta_i$. Let $x \in N$ have homogeneous decomposition $x = \sum_{\mu\in\mathbf{Z}} x_\mu$. Then $x = \sum_{i\in I} r_i z_i$ for some $r_i \in R$ with homogeneous components $(r_{i\lambda})_{\lambda\in\mathbf{Z}}$. We have

$$x = \sum_{\mu\in\mathbf{Z}} x_\mu = \sum_{i\in I}\Big(\sum_{\lambda\in\mathbf{Z}} r_{i\lambda}\Big) z_i = \sum_{\mu\in\mathbf{Z}}\Big(\sum_{\lambda+\delta_i=\mu} r_{i\lambda} z_i\Big)$$

The result follows from the uniqueness of the decomposition of x into its homogeneous components. ∎

Definition 4.1.4.

(a) An R-submodule N of M satisfying the above equivalent conditions is called a **graded R-submodule** of M.

(b) A **graded ideal** of a graded ring R is a graded submodule of the ring R considered as an R-module.

(c) A subring S of R becomes a **graded subring** provided it is a graded $\mathbf{Z}$-submodule of R.

(d) An R-subalgebra B of a graded R-algebra A is a **graded R-subalgebra** provided it is a graded R-submodule of A.

In order to manipulate these graded structures, we introduce graded homomorphisms.

Definition 4.1.5.

(a) A **graded homomorphism** between graded rings (algebras) R and S with gradings $(R_\mu)_{\mu\in\mathbf{Z}}$ and $(S_\mu)_{\mu\in\mathbf{Z}}$ is a ring (algebra) homomorphism $h : R \to S$ with $h(R_\mu) \subset S_\mu$ for all $\mu \in \mathbf{Z}$.

(b) Let M and N be graded R-modules with gradings $(M_\mu)_{\mu\in\mathbf{Z}}$ and $(N_\mu)_{\mu\in\mathbf{Z}}$. A **graded R-module homomorphism** of degree $\delta \in \mathbf{Z}$ between M and N is an R-module homomorphism $f : M \to N$ with $f(M_\mu) \subset N_{\mu+\delta} \forall \mu \in \mathbf{Z}$.

The canonical inclusion homomorphism of a graded sub R-module into a graded R-module is an example of a graded R-module homomorphism of degree 0. Graded R-modules are isomorphic if there exists a bijective graded homomorphism of degree 0 between them. In both cases, the corresponding statement holds for graded rings and algebras with the implicit degree requirement.

A graded R-module M is defined to be **free** if it has a basis $(m_i)_{i\in I}$ of homogeneous elements. Let λ_i be the degree of m_i and let $R(-\lambda_i)$ be the R-module R with the grading $R'_\mu = R_{\mu-\lambda_i}, \forall \mu \in \mathbf{Z}$. Let e_i be the identity element of R considered as an element of degree λ_i in $R(-\lambda_i)$. Then the R-module homomorphism $\mu : \oplus_{i\in I} R(-\lambda_i) \to M$ with $\mu(e_i) = m_i$ is a graded R-module isomorphism with $\mu(e_i) = m_i$.

Every graded R-module N is the graded homomorphic image of a graded free R-module. If $(n_i)_{i\in I}$ is a system of homogeneous generators of N with $\deg(n_i) = \mu_i$, then the R-module homomorphism

$$\rho : \oplus_{i\in I} R(-\mu_i) \to N$$

with $\rho(e_i) = n_i$ is a surjective graded R-module homomorphism of degree 0.

Graded quotient modules of a graded ring may also be defined. Let M be a graded R-module with grading $(M_\mu)_{\mu\in\mathbf{Z}}$ and N a graded submodule. Then we claim that $\left((M_\mu + N)/N\right)_{\mu\in\mathbf{Z}}$ forms a grading of the R-module M/N compatible with its R-module structure. By the second isomorphism theorem,

$$M_\mu + N/N \cong M_\mu/N_\mu = M_\mu/(N \cap M_\mu)$$

The first isomorphism theorem and the universal property of direct sums may be used to show

$$M/N \cong \oplus_{\mu\in\mathbf{Z}} M_\mu/N_\mu \cong \oplus_{\mu\in\mathbf{Z}} (M_\mu + N)/N$$

Also $R_\lambda(M_\mu + N) \subset R_\lambda M_\mu + N \subset M_{\mu+\lambda} + N$ implies that

$$R_\lambda\left((M_\mu + N)/N\right) \subset (M_{\lambda+\mu} + N)/N$$

Hence M/N is a graded R-module and the canonical homomorphism $M \to M/N$ is a graded R-module homomorphism of degree 0.

Similarly if I is a graded 2-sided ideal of a graded ring (algebra) R, then the ring (algebra) R/I with the quotient grading from its $\mathbf{Z}$-module (R-module) structure is a **graded quotient ring (algebra)**. Again the canonical homomorphism $R \to R/I$ is a graded ring (algebra) homomorphism.

The **symmetric algebra** of a n-dimensional vector space V over a field K is an example of a graded quotient algebra. It is defined to be $S(V) = T(V)/I$ where I is the ideal of $T(V)$ generated by all the elements of the form $x \otimes y - y \otimes x$ where $x, y \in V$. As the generators of this ideal are homogeneous of degree 2, I is a graded 2-sided ideal of $T(V)$. Hence $S(V)$ is a graded K-algebra with grading

$$S^k(V) = T^k(V)/I \cap T^k(V), k \geq 0$$

Note that as the elements of $S(V)$ are generated by $\varphi(V)$ where

$$\varphi : V = S^1(V) \to S(V)$$

is the canonical inclusion homomorphism and $\varphi(x)\varphi(y) = \varphi(y)\varphi(x)$, $S(V)$ is a commutative graded K-algebra. If $\{v_1, \cdots, v_n\}$ is a basis for V then

$$\{v_{i_1} \otimes \cdots \otimes v_{i_k} | i_1 \leq \cdots \leq i_k\}$$

is a basis for $S^k(V)$.

A second example of a graded quotient algebra is the **exterior algebra** of an n-dimensional vector space V over a field K. It is denoted $\Lambda(V) = T(V)/J$ where

J is the ideal of $T(V)$ generated by all the elements of the form $x \otimes x$ where $x \in V$. Again the generators of this ideal are homogeneous of degree 2 so that J is a graded 2-sided ideal of $T(V)$. $\Lambda(V)$ is a graded K-algebra with grading

$$\Lambda^k(V) = T^k(V)/J \cap T^k(V)$$

Let $\psi : V = \Lambda^1(V) \to \Lambda(V)$ be the canonical inclusion homomorphism. Since $\psi(x) \otimes \psi(y) = -\psi(y) \otimes \psi(x)$ for all $x, y \in V$ and $\psi(V)$ generates $\Lambda(V)$ as an algebra, $\Lambda(V)$ is an alternating graded K-algebra. We denote $x_1 \otimes \cdots \otimes x_k + J$ as $x_1 \wedge \cdots \wedge x_k$. Then a basis for $\Lambda^k(V)$ may be given by

$$\{v_{i_1} \wedge \cdots \wedge v_{i_k} | i_1 < \cdots < i_k\}$$

where $\{v_1, \cdots, v_n\}$ is a basis for V. Hence each $\Lambda^k(V)$ for $0 \leq k \leq n$ has dimension $\binom{n}{k}$ over K so that $\Lambda(V) = \oplus_{k=0}^{n} \Lambda^k(V)$ has dimension $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ over K.

We will need to have a generalized version of the first isomorphism theorem for graded R-modules.

Proposition 4.1.6. *Let R be a graded ring, M and N two graded R-modules and $u : M \to N$ a graded R-module homomorphism of degree δ. Then*

(i) $\mathrm{Im}(u)$ *is a graded submodule of N.*

(ii) $\mathrm{Ker}(u)$ *is a graded submodule of M.*

(iii) *If $\delta = 0$, then $\overline{u} : M/\mathrm{Ker}(u) \to \mathrm{Im}(u)$ is an isomorphism of graded R-modules.*

Proof:

(i) Since M is generated by homogeneous elements $(m_i)_{i \in I}$, $(u(m_i))_{i \in I}$ is a set of homogeneous generators of $\mathrm{Im}(u)$. By Proposition 4.1.3, $\mathrm{Im}(u)$ is a graded submodule of N.

(ii) Let $x = \sum_{\mu \in \mathbf{Z}} x_\mu \in \mathrm{Ker}(u)$ where $x_\mu \in M_\mu$. We want to show $x_\mu \in \mathrm{Ker}(u)$ for all $\mu \in \mathbf{Z}$. We have $\sum_{\mu \in \mathbf{Z}} u(x_\mu) = u(x) = 0$ where $u(x_\mu)$ is a homogeneous

element of degree $\mu + \delta$. Since $\deg(u(x_\mu)) \neq \deg(u(x_\lambda))$ when $\mu \neq \lambda$ we have $u(x_\mu) = 0\ \forall \mu \in \mathbf{Z}$ by the uniqueness of the decomposition. By Proposition 4.1.3(b), $\mathrm{Ker}(u)$ is a graded submodule of M.

(iii) By (ii), $M/\mathrm{Ker}(u)$ is a graded R-module with grading

$$\Big((M_\mu + \mathrm{Ker}(u))/\mathrm{Ker}(u)\Big)_{\mu \in \mathbf{Z}}.$$

So the canonical isomorphism of R-modules $\overline{u} : M/\mathrm{Ker}(u) \to \mathrm{Im}(u)$ maps $m_\mu + \mathrm{Ker}(u) \mapsto u(m_\mu) \in (\mathrm{Im}(u))_\mu$ and hence is an isomorphism of graded R-modules.

∎

Corollary 4.1.7. *Let A and B be two graded rings (algebras) and $u : A \to B$ a graded homomorphism of graded rings (algebras). Then $\mathrm{Im}(u)$ is a graded submodule of B, $\mathrm{Ker}(u)$ is a graded submodule of B and the canonical homomorphism $A/\mathrm{Ker}(u) \to \mathrm{Im}(u)$ is an isomorphism of graded rings (algebras).*

Section 4.2: Poincaré Series

Another important tool in the study of graded algebras such as S and S^G is the Poincaré series.

Definition 4.2.1. Let M be a graded module over the field K with the grading of M be $(M_n)_{n \geq 0}$ such that M_n has finite dimension over K. Then the **Poincaré series** for M is defined to be

$$P_M(t) = \sum_{n \geq 0} \dim(M_n) t^n$$

The following proposition is useful for computing Poincaré series.

Proposition 4.2.2. *Let M and N be graded K-modules with gradings $(M_n)_{n \geq 0}$, $(N_n)_{n \geq 0}$ as described above. Then*

a) $P_{M \oplus N}(t) = P_M(t) + P_N(t)$

b) $P_{M\otimes_K N}(t) = P_M(t)P_N(t)$

Proof: (a) follows immediately from

$$\dim_K(M+N)_n = \dim_K(M_n \oplus N_n) = \dim_K M_n + \dim_K N_n$$

For (b), the grading of $M \otimes_K N$ is given by $(M_\mu \otimes_K N_\lambda)_{n\geq 0}^{\mu+\lambda=n}$. So

$$\begin{aligned} P_{M\otimes_K N}(t) &= \sum_{\mu+\lambda=n, n\geq 0} \dim_K(M_\mu \otimes_K N_\lambda)t^n, \\ &= \sum_{\mu+\lambda=n, n\geq 0} (\dim_K M_\mu)(\dim_K N_\lambda)t^{\mu+\lambda}, \\ &= P_M(t)P_N(t) \end{aligned}$$

∎

The above proposition is sufficient to determine the Poincaré series of a graded commutative K-algebra with algebraically independent homogeneous generators. $K[X_1, \cdots, X_n]$ is an example of such a K-algebra. Later we will show that the symmetric algebra S and S^G may also be characterized this way.

Proposition 4.2.3. *Let $S = \oplus_{n\geq 0} S_n$ be a commutative graded K-algebra with homogeneous algebraically independent generators $\{x_1, \cdots, x_m\}$. Let $d_i = deg(x_i)$ for all i and assume $d_i > 0$. Then S_n is finite dimensional and*

$$P_S(t) = \prod_{i=1}^{m}(1-t^{d_i})^{-1}$$

Proof: We claim that $S \cong K[x_1] \otimes \cdots \otimes K[x_n]$.

The n-linear map $(f_1(x_1), \cdots, f_n(x_n)) \longmapsto f_1(x_1)\cdots f_n(x_n)$ from $\prod_{i=1}^{n} K[X_i]$ to S leads to a K-linear map

$$\varphi : f_1(x_1) \otimes \cdots \otimes f_n(x_n) \to f_1(x_1)\cdots f_n(x_n)$$

via the universal property of tensor products. By linearity, it is sufficient to check that φ is a K-algebra homomorphism on generators of the form

$$x_1^{k_1} \otimes \cdots \otimes x_n^{k_n}$$

From this verification, φ is a graded K-algebra homomorphism.

Define $\psi : S \to K[x_1] \otimes \cdots \otimes K[x_n]$ by extending $x_1^{k_1} \cdots x_n^{k_n} \to x_1^{k_1} \otimes \cdots \otimes x_n^{k_n}$ to a K-linear map. Again we need only check that ψ is a ring homomorphism on the generators $x_1^{k_1} \cdots x_n^{k_n}$ to see that ψ is also a graded K-algebra homomorphism.

As φ and ψ are inverses on their respective generators,

$$S \cong K[X_1] \otimes \cdots \otimes K[X_n]$$

are isomorphic under a graded K-algebra isomorphism.

The Poincaré series for $K[X_i]$ is $\sum_{n \geq 0} t^{nd_i} = (1 - t^{d_i})^{-1}$. Since S is isomorphic to $K[X_1] \otimes \cdots \otimes K[X_n]$, we have by the previous proposition and induction that $P_S(t) = \prod_{i=1}^{m} (1 - t^{d_i})^{-1}$. ∎

Corollary 4.2.4. *A set of homogeneous algebraically independent generators of S have degrees which are uniquely determined up to order.*

Proof: Examine the roots of the inverse of $P_S(t)$, $N_S(t) = \prod_{i=1}^{m} (1 - t^{d_i})$. Let $q \in \mathbf{N}$ and ρ be a primitive qth root of unity in $\mathbf{C}$. Then ρ is a root of $N_S(t)$ of multiplicity $m(q) = \sum_{q | d_i} 1$. There exists a minimal $N \in \mathbf{N}$ such that $m(q) = 0$ for $q \geq N$. Then by a recurrence relation, we may uniquely determine all d_i equal to q for $q < N$. ∎

We call S a graded polynomial K-algebra with characteristic degrees d_i. The number of degrees of S is the transcendence degree of S over K. Note that this number is the multiplicity of 1 as a root of the polynomial $N_S(t)$.

We can now compute the Poincaré series of $K[X_1, \cdots, X_n]$. Since it is generated by $X_1, \cdots, X_n$ over K, its series is $(1 - t)^{-n}$ and its transcendence degree over K is n. We will later show that the symmetric algebra $S(V)$ is isomorphic to $K[X_1, \cdots, X_n]$ and hence has the same Poincaré series and transcendence degree.

Next we look at the relationship of two graded polynomial K-algebras where one is a graded K-subalgebra of the other. Later $S(V)$ and $S(V)^G$ will be seen to satisfy the hypotheses of this proposition.

Proposition 4.2.5. *Let $S = \oplus_{n\geq 0} S_n$ and $R = \oplus_{n\geq 0} R_n$ be graded polynomial K-algebras where R is a graded subalgebra of S and the R-module S admits a finite base of homogeneous elements $z_1, \cdots, z_N$ of degrees $f_1, \cdots, f_n$. Then*

(i) *R and S have the same transcendence degree r over K.*

(ii) *Let $p_1, \cdots, p_r$ and $q_1, \cdots, q_r$ be the respective degrees of S and R. Then*

$$\prod_{i=1}^{r}(1-t^{q_i}) = (\sum_{j=1}^{N} t^{f_j}) \prod_{i=1}^{r}(1-t^{p_i})$$

(iii) $Np_1 \cdots p_r = q_1 \cdots q_r$

Proof: Let M be the graded K-module $\sum_{j=1}^{N} Kz_j$ and claim that $S \cong R \otimes_K M$. The bilinear map $(r, m) \mapsto rm$ from $R \times M$ to S produces a K-linear map $\varphi : r \otimes m \mapsto rm$. As $\varphi(R_\mu \otimes M_\lambda) \subset S_{\mu+\lambda}$, φ is a graded K-module homomorphism.

Define $\psi : S \to R \otimes_K M$ by

$$\sum_{i=1}^{n} r_i z_i \mapsto \sum_{i=1}^{n} r_i \otimes z_i$$

This is well-defined since S is a free R-module and it is easily verified to be a graded K-linear map. As φ and ψ are inverses on the generators $r \otimes z_i$ and rz_i of the K-modules $R \otimes_K M$ and S, they are isomorphic.

By Proposition 4.2.2, $P_S(t) = P_M(t)P_R(t)$ so that $N_R(t) = (\sum_{j=1}^{N} t^{f_j}) N_S(t)$. Then (i) follows since the multiplicity of 1 as a root of $N_R(t)$ and $N_S(t)$ must be the same. (ii) also follows from this calculation.

From (ii) we divide $(1-t)^r$ to obtain

$$\prod_{i=1}^{r}(1+t+t^2+\cdots+t^{q_i-1}) = (\sum_{j=1}^{N} t^{f_j}) \prod_{i=1}^{r}(1+t+\cdots+t^{p_i-1}).$$

By substituting $t = 1$, (iii) follows. ∎

Section 4.3: Graded Algebras $S(V), \Lambda(V)$ and S^G

Now that we have some background information on graded algebraic structures, we may examine more closely the example of the tensor algebra $T(V)$, the symmetric algebra $S = S(V)$ and the exterior algebra $\Lambda = \Lambda(V)$ where V is a vector space of dimension n over a field K of characteristic 0. It is possible to identify S with the graded K-algebra $K[X_1, \cdots, X_n]$.

Proposition 4.3.1. *$S(V)$ is isomorphic to $K[X_1, \cdots, X_n]$, the polynomial algebra of n commuting indeterminants over K.*

Proof: Let $\{v_1, \cdots, v_n\}$ be a basis for V. Let $\overline{\phi} : S^1(V) \to S(V)$ be the canonical inclusion homorphism. Then $\{\overline{\phi}(v_1) \cdots \overline{\phi}(v_n)\}$ is a set of algebraically independent homogeneous generators each of degree 1 for the commutative graded algebra $S(V)$. In the proof of Proposition 4.2.3, all commutative graded algebras with the same number of algebraically independent homogeneous generators of the same degree are shown to be isomorphic under a graded K-algebra isomorphism. Thus $S(V) \cong K[X_1, \cdots, X_n]$ where each X_i has degree 1.

Note that this result is consistent with the fact that

$$\{\overline{1}\} \cup \{\overline{v_{i_1} \otimes \cdots \otimes v_{i_n}} | i_1 \leq \cdots \leq i_n,\ n \in \mathbf{N}\}$$

is a basis for S over K.

It is possible to uniquely extend a linear mapping f from V to a K-algebra E to a K-algebra homomorphism $g : T(V) \to E$. In fact it is not hard to guess the definition of G. Let $\{v_1, \ldots, v_n\}$ be a basis for V. Intuitively g is defined as:

$$g(1) = 1$$

$$g(v_{i_1} \otimes \cdots \otimes v_{i_k}) = f(v_{i_1}) \cdots f(v_{i_k})$$

on the basis elements of $T(V)$. Similarly, if E is a commutative K-algebra f may be uniquely extended to a K-algebra homomorphism $h : S \to E$. Lastly, if E is an

alternating K-algebra, f may be uniquely extended to a K-algebra homomorphism $l : \Lambda \to E$. The following proposition formalizes the above remarks.

Proposition 4.3.2. *Let E be a K-algebra and $f : V \to E$ a K-linear map.*

(a) There exists a unique K-algebra homomorphism $g : T(V) \to E$ such that $f = g \circ \phi$ where $\phi : T^1(V) \to T(V)$ is the canonical injection.

(b) If E is commutative, there exists a unique K-algebra homomorphism $h : S \to E$ such that $f = g \circ \overline{\phi}$ where $\overline{\phi} : S^1(V) \to S(V)$ is the canonical injection.

(c) If E is alternating, there exists a unique K-algebra homomorphism $l : \Lambda(V) \to E$ such that $f = l \circ \psi$ where $\psi : \Lambda^1(V) \to \Lambda(V)$ is the canonical injection.

(d) If, in addition, E is graded with grading $(E_\lambda)_{\lambda \in \mathbf{Z}}$ and $f(V) \subset E_1$ then g, h, l respectively are graded K-algebra homomorphisms.

Proof:

(a) For all $n > 0$, there exists a n-linear map $(x_1, \cdots, x_n) \mapsto f(x_1, \cdots, x_n)$ so that by the universal property of tensor products there exists a K-linear map $g_n : T^n(V) \to E$ with $g_n(x_1 \otimes \cdots \otimes x_n) = f(x_1) \cdots f(x_n)$. Also define $g_0 : T^0(V) \to E$ such that $g_0(1) = 1$. By the universal property of direct sums there exists a unique K-linear map $g : T(V) \to E$ with $g|_{T^n(V)} = g_n$ for all $n \geq 0$. Then $g \circ \phi = g_1 = f$ and we need only verify that g is a ring homomorphism. By construction, $g(1) = 1$, so by linearity we need only show $g(uv) = g(u)g(v)$ for $u = x_1 \otimes \cdots \otimes x_p$ and $v = x_{p+1} \otimes \cdots \otimes x_{p+q}$. This follows directly from the definition of g and of multiplication in $T(V)$.

Let V have a basis $\{v_1, \cdots, v_n\}$. Then $T(V)$ has a basis

$$\{1\} \cup \{v_{i_1} \otimes \cdots \otimes v_{i_k} | k \in \mathbf{N}\}.$$

Then

$$g(v_{i_1} \otimes \cdots \otimes v_{i_k}) = f(v_{i_1}) \cdots f(v_{i_k})$$

$$g(1) = 1$$

(b) To obtain the required homomorphism h, it suffices to show $I \subset \mathrm{Ker}(g)$. Then $g = h \circ q$ where $q : T(V) \to T(V)/I$ is the canonical quotient homomorphism. Since $\overline{\phi} = q \circ \phi$, we will then have $h \circ \overline{\phi} = h \circ q \circ \phi = g \circ \phi = f$ as required. The kernel of g is a 2-sided ideal so that it suffices to show the generators of I are in $\mathrm{Ker}(g)$. But $g \circ \phi = f$ implies $g(x \otimes y - y \otimes x) = f(x)f(y) - f(y)f(x) = 0$ so that $I \subset \mathrm{Ker}(g)$. Hence h exists as required. The uniqueness of h follows from that of g and the universal property of quotient homomorphisms.

(c) Following the pattern of (b), to find the homomorphism l, we need only show that $J \subset \mathrm{Ker}(g)$. Let $q' : T(V) \to T(V)/J$ be the canonical quotient homomorphism. Then $\psi = q' \circ \phi$ implies that $l \circ \psi = l \circ q' \circ \phi = g \circ \phi = f$. Again it suffices to show that the generators of J are in Ker(g). But $g \circ \phi = f$ implies that $g(x \otimes x) = f(x)f(x) = 0$ so that $J \subset \mathrm{Ker}(g)$. Hence l exists as required. Again the uniqueness of l follows from that of g.

(d) If E is graded and $f(V) \subset E_1$, then $g(T_n(V)) \subset E_n$ by definition of g on the generators of $T_n(V)$. In this case, g is a graded K-algebra homomorphism. Similarly h and l are graded K-algebra homomorphisms as they are the homomorphisms induced by g on $T(V)/I$ and $T(V)/J$ respectively.

Proposition 4.3.3. *Let $u, v : V \to V$ be K-linear maps.*

(a) *There exists a unique graded K-algebra homomorphism $S(u) : S(V) \to S(V)$ such that the diagram*

$$\begin{array}{ccc} V & \xrightarrow{u} & V \\ {\scriptstyle \overline{\phi}}\downarrow & & \downarrow{\scriptstyle \overline{\phi}} \\ S(V) & \xrightarrow[S(u)]{} & S(V) \end{array}$$

commutes.

(b) There exists a unique graded K-algebra homomorphism $\Lambda(u) : \Lambda(V) \to \Lambda(V)$

such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{u} & V \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ \Lambda(V) & \xrightarrow[\Lambda(u)]{} & \Lambda(V) \end{array}$$

commutes.

(c) $S(1_V) = 1_S, \Lambda(1_V) = 1_\Lambda$

(d) Let $u, v \in GL(V)$. Then

$$S(u \circ v) = S(u) \circ S(v), [S(u)]^{-1} = S(u^{-1}),$$

$$\Lambda(u \circ v) = \Lambda(u) \circ \Lambda(v), [\Lambda(u)]^{-1} = \Lambda(u^{-1})$$

Proof:

(a) The existence and uniqueness of $S(u)$ follow from Proposition 4.3.2 applied to $E = S(V)$ and $f = \overline{\phi} \circ u : V \to S(V)$. Since $u(V) \subset S^1(V) = V$, $S(u)$ is a graded algebra homomorphism.

(b) Analogous to (a).

(c) 1_S is a graded K-algebra homomorphism which makes the diagram

$$\begin{array}{ccc} V & \xrightarrow{1_V} & V \\ {\scriptstyle\overline{\phi}}\downarrow & & \downarrow{\scriptstyle\overline{\phi}} \\ S(V) & \xrightarrow[1_S]{} & S(V) \end{array}$$

commute. By uniqueness, $S(1_V) = 1_S$. Similarly, $\Lambda(1_V) = 1_\Lambda$.

(d) $S(u) \circ S(v) : S \to S$ is a graded K-algebra homomorphism which makes the diagram

$$\begin{array}{ccc} V & \xrightarrow{u \circ v} & V \\ {\scriptstyle\overline{\phi}}\downarrow & & \downarrow{\scriptstyle\overline{\phi}} \\ S(V) & \xrightarrow[S(u) \circ S(v)]{} & S(V) \end{array}$$

commute. By uniqueness, $S(u \circ v) = S(u) \circ S(v)$. The other results follow easily.

Let G be a finite subgroup of GL(V) . We may now use the second proposition to show that the action of G on V may be uniquely extended to an action of G on S and

Λ. Each element $g \in G$ induces an automorphism $\rho(g) \in GL(V)$ with the properties $\rho(1_V) = 1_V, \rho(gh) = \rho(g) \circ \rho(h)$. Hence $S(\rho(g))$ is the unique automorphism of S extending $\rho(g)$ to S. It satisfies

$$S \circ \rho(1_S) = 1_S, S \circ \rho(gh) = S(\rho(g)) \circ S(\rho(h))$$

Hence the action of G may be extended uniquely from V to S. Similarly the action of G may be extended uniquely from V to Λ. For simplicity, we will denote the automorphisms induced by G on S and Λ as $s \mapsto g \cdot s, e \mapsto g \cdot e$ respectively.

Consider the invariants of the action of G on S:

$$S^G = \{s \in S | g \cdot s = s \; \forall g \in G\}$$

S^G forms a K-subalgebra of S. In fact S^G is a graded K-subalgebra of S. More generally, we have the following proposition.

Proposition 4.3.4. *Let A be a graded K- vector space and let H be a finite group which acts homogeneously on A. Then A^H is a graded K subspace of A.*

Proof: Suppose $a \in A^H$, where $a = \sum_{m \geq 0} a_m, a_m \in A_m$. Then for $h \in H$, $h \cdot a = \sum_{m \geq 0} h \cdot a_m$. Since h induces a graded vector space automorphism, $h \cdot a_m \in A_m$ for all $m \geq 0$. This implies that A^H is a graded K-subspace of A with grading $\{A^H \cap A_m\}_{m \geq 0}$.

The next theorem is the first stage of our characterization of the graded algebra S and its graded subalgebra $R = S^G$. It summarizes the results of a few propositions that follow.

Theorem 4.3.5. *S is a finitely generated graded S^G-module of rank $|G|$.*

The fact that S is a graded S^G-module follows from the above discussion about the grading of S^G. The rest of the statement holds more generally for any finitely generated integral K-algebra on which the action of a finite group has been defined as will be shown in the next few propositions.

Proposition 4.3.6. *Let A be a finitely generated K-algebra and H be a finite group which acts on A. Then A is a finitely generated A^H-module.*

Proof: We first show that every element of A is integral over A^H. Let $x \in A$ and let $\{x_1, \cdots, x_n\}$ be the orbit of x under H. Then for all $h \in H$, there exists $\pi_h \in S_n$ such that $h \cdot x_i = x_{\pi_h(i)}$ for $1 \leq i \leq n$. Thus the elementary functions of the x_i belong to A^H so that x is a root of $\prod_{i=1}^{n}(X - x_i)$, a monic polynomial in $A^H[X]$. Hence A is integral over A^H.

Let $\{a_j\}_{j=1}^{m}$ be a set of generators of A over K. We show that $A = A^H[a_1, \cdots, a_m]$ is finitely generated over A^H by induction on m. For a integral over an algebra B, $a^n + \beta_{n-1}a^{n-1} + \cdots + \beta_0 = 0$ for some $\beta_i \in B, n \in \mathbf{N}$ so that the K-algebra $B[a]$ is generated over B by $1, a, \cdots, a^{n-1}$. Hence $A^H[a_i]$ is finitely generated over A^H for all $i = 1 \cdots, m$. By the inductive hypothesis, $A^H[a_1, \cdots, a_{n-1}]$ is finitely generated over A^H so that $A = A^H[a_1, \cdots, a_n] = A^H[a_1, \cdots, a_{n-1}][a_n]$ is finitely generated over A^H.

Thus in our case S is a finitely generated S^G-module. To determine the rank of the S^G module S, we will examine their respective quotient fields and use some Galois Theory.

Returning to the notation of the previous proposition, we will assume in addition that A is integral. Let N and L be the respective quotient fields of A and A^H. Since a K-algebra automorphism of an integral K-algebra can be uniquely extended to an automorphism of its field of fractions, the action of H on A can be extended to an action of H on N via $h \cdot \frac{a}{b} = \frac{h \cdot a}{h \cdot b}$ where $h \in H, a \in A, b \in A - \{0\}$.

Proposition 4.3.7. *L is the fixed field of N under the action of H and $[N : L] = |H|$.*

Proof: We show $N^H = L$.

For $\frac{a}{b} \in L$ and $h \in H$, $h\left(\frac{a}{b}\right) = \frac{h(a)}{h(b)} = \frac{a}{b}$ so that $L \subset N^H$.

To prove $N^H \subset L$, we first show that every element of N may be expressed as $\frac{a}{b}$ where $a \in A, b \in A^H - \{0\}$.

Let $\frac{c}{d} \in N$. Then $\frac{c}{d} = \frac{c \prod_{h \neq 1} h \cdot d}{\prod_{h \in H} h \cdot d}$ where $\prod_{h \in H} h \cdot d \in A^H - \{0\}$.

Let $\frac{a}{b} \in N^H$. We may assume $a \in A$ and $b \in A^H - \{0\}$. Then for $h \in H$, $\frac{h(a)}{b} = h\left(\frac{a}{b}\right) = \frac{a}{b}$. Hence $a \in A^H$ and $N^H \subset L$.

Since H is a finite subgroup of Aut(N) and $L = N^H$ is the fixed field of N, N/L is a Galois extension with Galois group H so that $[N : L] = |H|$.

Proposition 4.3.8. *With the above notation and $B = A^H$, $\text{rank}_B A = [N : L]$.*

Proof: Let $\{z_1, \cdots, z_m\}$ be a maximal linearly independent subset of A over B. Let $\phi : A \to N$ be the canonical inclusion homomorphism with $\phi(a) = \frac{a}{1}$. We claim that $\{\phi(z_1), \cdots, \phi(z_m)\}$ is a linearly independent set in N over L.

Set

$$\sum_{i=1}^{m} \left(\frac{a_i}{b_i}\right)\phi(z_i) = 0$$

for $a_i \in A^H, b_i \in A^H - \{0\}$. Multiply the equation by $b_1 \cdots b_m \in A^H - \{0\}$. Then $\sum_{i=1}^m t_i a_i \phi(z_i) = 0$ where $t_i = \prod_{j \neq i} b_j \in A^H - \{0\}$ so that $\phi(\sum_{i=1}^m t_i a_i z_i) = 0$. But since ϕ is injective, $\sum_{i=1}^m t_i a_i z_i = 0$. Then for all i, $t_i a_i = 0$ and $t_i \neq 0$ imply that $\frac{a_i}{b_i} = 0$. Thus $\text{rank}_B A \leq [N : L]$.

Let $\left\{\frac{z_1}{t_1}, \cdots, \frac{z_n}{t_n}\right\}$ be a basis of N/L where $z_i \in A, t_i \in B - \{0\}$. Set $\sum_{i=1}^n \alpha_i z_i = 0$ with $\alpha_i \in B$. Divide the equation by $\prod_{j=1}^n t_j$. Then $\sum_{i=1}^n \frac{\alpha_i}{\prod_{i \neq i} t_j} \frac{z_i}{t_i} = 0$ so that $\frac{\alpha_i}{\prod_{i \neq i} t_j} = 0$ for all i implying that $\alpha_i = 0$ for all i. Hence $[N : L] \leq \text{rank}_B A$.

Combining all the above results, we have shown that S is a graded S^G module with rank equal to the cardinality of G. We have completed the proof of the theorem.

CHAPTER 5

Degrees of a Finite Reflection Group

Throughout this chapter V will denote an n-dimensional vector space over a field K of characteristic zero with symmetric algebra $S = S(V)$ and exterior algebra $\Lambda = \Lambda(V)$. Unless otherwise indicated G will be a finite reflection subgroup of $GL(V)$. The material discussed follows the exposition in [H2] and [B3].

Section 5.1: Chevalley's Theorem

The goal of this section is to prove that S^G is a graded polynomial subalgebra of S, a result due to Chevalley. Combining this with the result of the previous chapter will allow us to characterize S as a free graded S^G-module of rank $|G|$. For convenience, we will identify S with $K[X_1, \cdots, X_n]$ where $\{X_1, \cdots, X_n\}$ is a basis for V and we will often denote S^G by R.

Definition 5.1.1. The **averaging operator** $\alpha : S \to R$ is the graded K-algebra homomorphism defined by $\alpha(s) = \frac{1}{|G|}\sum_{g \in G} S(g)(s)$ for $s \in S$. Note that α is a projection from S onto R.

The first step in proving Chevalley's theorem is to show that $R = S^G$ is finitely generated as a K-algebra. Let the grading of R be given by $(R^n)_{n \geq 0}$. Denote $R^+ = \oplus R^n$ and let $I = SR^+$ be the ideal generated by R^+ in S.

Proposition 5.1.2. *$R = S^G$ is a finitely generated K-algebra.*

Proof: $I = SR^+$ is clearly a graded ideal of S. In fact it is also finitely generated by Hilbert's Basis Theorem since $S \cong K[X_1, \cdots, X_n]$. Let $f_1, \cdots, f_r$ be homogeneous elements of R^+ which generate I as an ideal. We claim that R is generated as a K-algebra by $1, f_1, \cdots, f_r$. It is sufficient to prove this claim for a homogeneous element f of R by induction on its degree.

For $\deg f = 0$, the result is clear.

Let $\deg f > 0$. Then $f \in R^+$ so that $f = \sum_{i=1}^{r} s_i f_i$ for some $s_i \in S$. By examining only the terms of $\deg f$ we may assume $\deg f = \deg s_i + \deg f_i$ for all i. Applying the averaging operator, we obtain $f = \sum_{i=1}^{r} \alpha(s_i) f_i$. But each $\alpha(s_i) \in R$ has $\deg \alpha(s_i) = \deg s_i < \deg f$. By the inductive hypothesis, $\alpha(s_i)$ may be expressed as a polynomial in $f_1, \cdots, f_r$. The result follows. ∎

The following technical lemma is a crucial step in the proof of Chevalley's Theorem. The result hinges on the fact that G is a finite reflection group.

Lemma 5.1.3. *Let $f_1, \cdots, f_r \in R$ with $f_1 \notin R[f_2, \cdots, f_r]$, the ideal of R generated by $f_2, \cdots, f_r$. Suppose that $h_1, \cdots, h_r$ are homogeneous elements of S satisfying*

$$f_1 h_1 + \cdots + f_r h_r = 0 \tag{1}$$

Then $h_1 \in I := SR^+$.

Proof: Note that $f_1 \notin S[f_2, \cdots, f_r]$. Otherwise, $f_1 = \sum_{i=2}^{r} k_i f_i$ where $k_i \in S$. Applying the averaging map $\alpha : S \to R$

$$f_1 = \alpha(f_1) = \sum_{i=2}^{r} \alpha(k_i) f_i \in R[f_2, \cdots, f_r]$$

We prove the statement using induction on the degree of h_1. If $\deg h_1 = 0$, h_1 is a constant. Suppose $h_1 \neq 0$. Then (1) shows that $f_1 \in S[f_2, \cdots, f_r]$ contradicting the hypothesis. Hence $h_1 = 0 \in I$.

Suppose $\deg h_1 > 0$. Since I is a graded 2-sided ideal of the graded K-algebra S, S/I is a graded quotient algebra. Consider the natural action of G on S/I. We wish to show that G acts trivially on $h_1 + I$. Since G is generated by reflections it is sufficient to consider the action of a reflection g on S.

For all $x \in V$, $g \cdot x - x \in Kv$ for some non-zero element $v \in V$. Then as S is generated by V, $S(g)(y) - y \in Sv$ for all $y \in S$. But then

$$S(g)(y) - y = k(y)v \tag{2}$$

where $k : S \to S$ is a graded R-module homomorphism of degree -1. Applying the graded R-module map $S(g) - 1_S$ to (1) and using (2) we obtain

$$(f_1 k(h_1) + \cdots + f_r k(h_r))v = 0 \tag{3}$$

Since $v \neq 0$ and S is an integral domain,

$$f_1 k(h_1) + \cdots + f_r k(h_r) = 0$$

As $k(h_1)$ is a homogeneous element of S of degree less than that of h_1, we have by inductive hypothesis that $k(h_1) \in I$. By (2) $S(g)(h_1) - h_1 \in I$. Hence we obtain $S(g)(h_1) \equiv h_1 \mod I$ for all $g \in G$.

Now we apply the averaging map to $h_1 + I$ resulting in $\alpha(h_1) \equiv h_1 \mod I$. But $\alpha(h_1) \in R_+ \subset I$ so that $h_1 \in I$ as required. ∎

To prove Chevalley's Theorem, we will work with formal partial derivatives of polynomials in $K[X_1, \cdots, X_n]$ where S and $K[X_1, \cdots, X_n]$ are identified under an isomorphism. The following identity due to Euler for a homogeneous polynomial f will be useful

$$\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = (\deg f) f$$

Theorem 5.1.4. *(Chevalley) S^G is a graded polynomial K-algebra of transcendental degree n. In particular, S^G is generated as a K-algebra by n homogeneous algebraically independent elements of positive degree and 1.*

Proof: Consider the ideal $I = SR^+$ in S. As we have seen before, we can choose a minimal generating set $\{f_1, \cdots, f_r\}$ for I of homogeneous elements which with 1 will generate R as a K-algebra. We will show that this set is algebraically independent. By Proposition 4.2.5(i), this will prove that r=n.

Suppose $\{f_1, \cdots, f_r\}$ is not an algebraically independent set. Identifying S with $K[X_1, \cdots, X_n]$, there exists a polynomial $h \in K[X_1, \cdots, X_n]$ such that

$$h(f_1, \cdots, f_r) = 0 \tag{1}$$

Choose h to be a homogeneous polynomial of minimal degree d. This choice is possible because of the linear independence of the monomials

$$\{X_1^{k_1} \cdots X_r^{k_r} | \sum_{i=1}^{r} k_i = k\}.$$

Differentiate (1) with respect to each x_k using the chain rule. Hence we have for each $k = 1, \cdots, n$,

$$\sum_{i=1}^{r} h_i \frac{\partial f_i}{\partial x_k} = 0 \text{ where } h_i = \frac{\partial h}{\partial y_i}(f_1, \cdots, f_r) \tag{2}$$

Then if f_i has degree d_i, each h_i is a homogeneous element of R of degree $d - d_i$ and $\frac{\partial f_i}{\partial x_k}$ is a homogeneous element of S of degree $d_i - 1$. We want to apply the lemma to equation (2) but we must first rearrange the expression so that the hypotheses are satisfied. If necessary, renumber the h_i so that $\{h_1, \cdots, h_m\}$ forms a minimal generating set for the ideal $R[h_1, \cdots, h_r]$ in R. Then for $j > m$,

$$h_j = \sum_{i=1}^{m} g_{ij} h_i \text{ for some } g_{ij} \in R \tag{3}$$

Again by the linear independence of terms of the same total degree, we may assume each g_{ij} is homogeneous of degree $d_i - d_j$ in R.

Substituting (3) into (2) we obtain

$$\sum_{i=1}^{m} h_i \left(\frac{\partial f_i}{\partial x_k} + \sum_{j=m+1}^{n} g_{ij} \frac{\partial f_j}{\partial x_k}\right) = 0 \tag{4}$$

for each $k = 1, \cdots, n$. Letting the term in parentheses be p_i, we note that p_i is a homogeneous element of S of degree $d_i - 1$. Now we may apply the lemma to (4) so that $p_1 \in I$. This means

$$\frac{\partial f_1}{\partial x_k} + \sum_{j=m+1}^{r} g_{1j} \frac{\partial f_j}{\partial x_k} = \sum_{i=1}^{r} q_{ik} f_i \tag{5}$$

for each $k = 1, \cdots, n$ where $q_{ik} \in S$. Multiplying by x_k, summing over $k = 1, \cdots, r$ and using Euler's identity, we get

$$d_1 f_1 + \sum_{j=m+1}^{r} g_{1j} d_j f_j = \sum_{i=1}^{r} r_i f_i \tag{6}$$

where $r_i = \sum_{k=1}^{n} x_k q_{ik} \in S$ has positive degree. Identifying only the terms of degree d_1, we note that $f_1 \in S[f_2, \cdots, f_r]$ which contradicts the minimality of the generating set chosen for I. Hence $f_1, \cdots, f_r$ are algebraically independent. ∎

From this theorem, we now know that R is a graded K-algebra of polynomials. Homogeneous algebraically independent generators of R will be called **basic invariants**. Note that by Corollary 4.2.4, any set of algebraically independent generators of R will have the same set of degrees.

Definition 5.1.5. The degrees $d_1 \leq \cdots \leq d_n$ of a finite reflection group G are the **characteristic degrees** of S^G.

Chevalley's Theorem enables us to show that S is a free R-module. To do this we will compare the R-module S with the vector space S/R^+S.

Proposition 5.1.6. *S is a free graded S^G-module of rank $|G|$.*

Proof: Since S/R_+S is a graded R-module it is also a graded $R_0 = K$ - vector space. Let $I = R_+S$ and let $\{g_\alpha\}_{\alpha \in \Gamma}$ be a set of homogeneous elements of S whose cosets $\{g_\alpha + I\}_{\alpha \in \Gamma}$ span S/I over K. We claim that the set $\{g_\alpha\}_{\alpha \in \Gamma}$ spans S over R.

Let T be the graded R-submodule of S generated by $\{g_\alpha\}_{\alpha \in \Gamma}$. Let the gradings of S and T be given by $\{S^d\}_{d \geq 0}$ and $\{T^d\}_{d \geq 0}$ respectively. It is sufficient to show by induction that $S^d = T^d$ for all $d \geq 0$.

Since $S^0 \cap I = \{0\}$, we find that $S^0 = T^0$.

Let $f \in S^d$ with $d > 0$. Then

$$f + I = \sum_{\alpha} c_\alpha (g_\alpha + I)$$

for some $c_\alpha \in K$. Hence

$$f = \sum_{\alpha} c_\alpha g_\alpha + \sum_{\beta} h_\beta f_\beta$$

where $h_\beta \in R_+, f_\beta \in S$. Discarding redundant terms, we may assume that h_β and f_β are homogeneous with $\deg(h_\beta) + \deg(f_\beta) = \deg(f)$ for all β. But then by the inductive hypothesis, we find $f_\beta \in S^{d_\beta} = T^{d_\beta}$ where $\deg(f_\beta) = d_\beta < d$ for each β. Hence $f \in T^d$ and by induction S is spanned by $\{g_\alpha\}_{\alpha \in \Gamma}$ over R.

Suppose $g_1, \ldots, g_m$ are homogeneous elements of S whose cosets $\{g_i + I\}_{i=1}^m$ are linearly independent over K. We use induction on m to show that $\{g_i\}_{i=1}^m$ is a linearly independent set over R. Let $\sum_{i=1}^m f_i g_i = 0$ with $f_i \in R$. Without loss of generality, we may assume that the f_i's are homogeneous elements of R with $\deg(f_i) + \deg(g_i) = c$ where c is a constant. Since $g_1 \notin I$, we find that $f_1 \in R[f_2, \ldots, f_n]$ in contradiction with Lemma 5.1.3. Hence $f_1 = \sum_{i=2}^m h_i f_i$ where we may assume the h_i are homogeneous elements of R with $\deg(h_i) + \deg(f_i) = \deg(f_1)$. We have

$$\sum_{i=2}^{m} (h_i g_1 + g_i) f_i = 0.$$

To apply the inductive hypothesis, we need only show that $\{h_i g_1 + g_i\}_{i=2}^m$ is a set of homogeneous elements of S whose cosets are linearly independent over R. From this it will follow that $f_2, \ldots, f_m$ are all 0 and hence $f_1 = 0$ so that $\{g_1, \ldots, g_m\}$ is a linearly independent set over R.

Indeed $\deg(h_i g_1) = \deg(h_i) + \deg(g_1) = \deg(f_1) - \deg(f_i) + \deg(g_1) = c - \deg(f_i) = \deg(g_i)$ implies that $\{h_i g_1 + g_i\}_{i=2}^m$ is a set of homogeneous elements of S. Since $h_i g_1 + g_i + I = \alpha_i g_1 + g_i + I$ where

$$\alpha_i = \begin{cases} 0, & \text{if } \deg(h_i) = 0; \\ h_i, & \text{if } \deg(h_i) \neq 0, \end{cases}$$

we find that

$$\sum_{i=2}^{m} c_i (h_i g_1 + g_i + I) = I$$

for some $c_i \in K$ implies that

$$\left(\sum_{i=2}^{m} c_i \alpha_i \right) (g_i + I) + \sum_{i=2}^{m} c_i (g_i + I) = I$$

so that $c_i = 0$ for all i. Hence $\{h_i g_1 + g_i + I\}_{i=2}^m$ is a linearly independent set over K so that $\{g_i\}_{i=1}^m$ is a linearly independent set over R.

As the rest of the proposition was already proven in Theorem 4.3.5, we have our result. ∎

Section 5.2: Poincaré Series for S^G

Since S is isomorphic to $K[X_1, \cdots, X_n]$, it too has a Poincaré series $(1-t)^{-n}$. We wish to derive a formula for the Poincaré series of R. A few lemmas are developed here to prove the formula. They are stated separately as they will be used again later on.

Lemma 5.2.1. *Let H be a finite group. Let E be any finite dimensional H-module over K. Then the dimension of the space of H-invariants in E is the average of the traces of the linear operators induced by elements of the group. That is:*

$$\dim_K E^H = \frac{1}{|H|} \sum_{h \in H} \mathrm{tr}(h).$$

Proof: We use the averaging operator $z = \frac{1}{|H|} \sum_{h \in H} h$ and show that this is a projection from E to E^H with trace equal to the dimension of E^H.

Note that $h \cdot z = z$. Then

$$z^2 = \frac{1}{|H|} \sum_{h \in H} h \cdot z = z$$

so that z is idempotent. But then z is diagonalizable since its minimal polynomial divides $z^2 - z$ and hence has distinct roots.

Decompose E into its eigenspaces $E = E_0 \oplus E_1$. The trace of the operator z is then the dimension of E_1. We need only show that $E^H = E_1$.

Suppose $e \in E_1$. Then for each $h \in H$,

$$h \cdot e = h \cdot z \cdot e = z \cdot e = e$$

so that $e \in E^H$. On the other hand, $e \in E^H$ implies that

$$z \cdot e = \frac{1}{|H|} \sum_{h \in H} h \cdot e = \frac{1}{|H|} \sum_{h \in H} e = e$$

and so $e \in E_1$. Hence,

$$\frac{1}{|H|} \sum_{h \in H} \mathrm{tr}(h) = \mathrm{tr}(z) = \dim_K(E^H). \quad \blacksquare$$

Proposition 5.2.2. *Let G be a finite subgroup of $GL(V)$. Let $g \in G$ with eigenvalues $\lambda_1(g), \cdots, \lambda_n(g)$. Denote the canonical extensions of g to $S^m(V)$ and $\Lambda^p(V)$ as $S^m(g)$, $\Lambda^p(g)$ respectively. Then*

(i) $\sum_{m=0}^{\infty} \mathrm{tr}(S^m(g))t^m = [\det(1 - gt)]^{-1} = \prod_{i=1}^{n}(1 - \lambda_i(g)t)^{-1}$.

(ii) $\sum_{p=0}^{n} \mathrm{tr}(\Lambda^p(g))t^p = \det(1 + gt) = \prod_{i=1}^{n}(1 + \lambda_i(g)t)$.

(iii) $\mathrm{tr}(\Lambda^p(g)) = \sigma_p(\lambda_1(g), \cdots, \lambda_n(g))$ *where σ_p is the pth symmetric function in n variables.*

Proof: Without loss of generality, we may assume that K is algebraically closed by extending the base field. Then every element of G is diagonalizable as an element of finite order in $GL(V)$. Choose a basis $\{x_1, \cdots, x_n\}$ for V of eigenvectors of g corresponding to the eigenvalues $\lambda_1, \cdots, \lambda_n$. Identify S with $K[x_1, \cdots, x_n]$.

(i) The matrix of $S^m(g)$ with respect to the basis $\{x_1^{k_1} \cdots x_n^{k_n} | \sum_{i=1}^{n} k_i = m\}$ ordered lexicographically has diagonal elements $(\lambda_1^{k_1} \cdots \lambda_n^{k_n})_{\sum_{i=1}^{n} k_i = m}$ Then

$$\begin{aligned}
\sum_{m=0}^{\infty} \mathrm{tr}(S^m(g))t^m &= \sum_{m=0}^{\infty} \sum_{k_1 + \cdots + k_n = m} \lambda_1^{k_1} \cdots \lambda_n^{k_n} \\
&= (\sum_{k_1=0}^{\infty} \lambda_1^{k_1} t^{k_1}) \cdots (\sum_{k_n=0}^{\infty} \lambda_1^{k_n} t^{k_n}). \\
&= (1 - \lambda_1 t)^{-1} \cdots (1 - \lambda_n t)^{-1} \\
&= [\det(1 - gt)]^{-1}
\end{aligned}$$

(ii) & (iii) The matrix of $\Lambda^p(g)$ with respect to the basis

$$\{x_{i_1} \cdots x_{i_p} | i_1 < \cdots < i_p\}$$

ordered lexicographically has diagonal entries $\lambda_{i_1} \cdots \lambda_{i_p}$ with $i_1 < \cdots < i_p$. Then $\mathrm{tr}(\Lambda^p(g)) = \sigma_p(\lambda_1(g), \cdots, \lambda_n(g))$. But this is also the coefficient of t^p in $\prod_{i=1}^{n}(1 + \lambda_i(g)t)$. Hence we have proven (ii) and (iii). ∎

Proposition 5.2.3. *Let G be a finite reflection subgroup of $GL(V)$. Let $R = S^G$. Then the Poincaré series for R is*

$$\frac{1}{|G|}\sum_{g\in G}[\det(1 - gt)]^{-1}.$$

Proof: Since the averaging operator $\alpha = \frac{1}{|G|}\sum_{g\in G} g$ is a graded R-module projection from S onto R, the operator $\alpha_m = \frac{1}{|G|}\sum_{g\in G} S^m(g)$ is a projection of S^m onto R^m. By the first lemma, the dimension of R^m is the trace of α_m. But then,

$$\begin{aligned} P_R(t) &= \sum_{m\geq 0} \dim(R^m)t^m \\ &= \sum_{m\geq 0} \mathrm{tr}(\alpha_m)t^m \\ &= \frac{1}{|G|}\sum_{g\in G}\sum_{m\geq 0} \mathrm{tr}(S^m(g))t^m \\ &= \frac{1}{|G|}\sum_{g\in G}[\det(1 - gt)]^{-1} \end{aligned}$$

∎

Section 5.3: Properties of the Degrees of G

The degrees of G have some remarkable properties which reflect information about G.

Proposition 5.3.1. *The product of the degrees of G is $|G|$.*

Proof: From 5.1.6, we know that $\mathrm{rank}_R(S) = |G|$. Since $S \cong K[X_1, \cdots, X_n]$, the characteristic degrees of S are all 1. Hence the result follows from Proposition 4.2.5(iii). ∎

Proposition 5.3.2. *Let H be the set of reflections in G distinct from the identity. Then*

$$|H| = \sum_{i=1}^{n}(d_i - 1)$$

where $d_1, \cdots, d_n$ are the degrees of G.

Proof: Assume K is algebraically closed. Every element of G is diagonalizable with eigenvalues $\lambda_1(g), \cdots, \lambda_n(g)$. Then $g = 1$ if and only if all eigenvalues of g are 1 and $g \in H$ if and only if 1 is an eigenvalue of g of multiplicity n-1. That means that $\det(1 - gt)$ is divisible by $(1-t)^{n-1}$ if and only if $g \in H \cup \{1\}$. Denote the distinct eigenvalue of $g \in H$ by $\lambda(g)$. Then equate the two expressions for the Poincaré series of R given by Propositions 4.2.3 and 5.2.3. We get

$$\prod_{i=1}^{n}(1-t^{d_i})^{-1} = \frac{1}{|G|}\sum_{g\in G}[\det(1-gt)]^{-1}$$

Multiplying through by $(1-t)^n$ we obtain:

$$\frac{1}{\prod_{i=1}^{n}(1+t+\cdots+t^{d_i-1})} = \frac{1}{|G|}\Big[1+\sum_{g\in H}\frac{(1-t)}{(1-\lambda(g)t)}+\sum_{g\notin H, g\neq 1}\frac{(1-t)^n}{\det(1-gt)}\Big] \quad (1)$$

Set $t = 1$ and take reciprocals of both sides to show $|G| = d_1 \cdots d_n$ as we observed in Proposition 5.3.1. Formally differentiate both sides of (1) to get

$$\Big(\prod_{i=1}^{n}\frac{1}{1+t+\cdots+t^{d_i-1}}\Big)\Big(\sum_{i=1}^{n}-\frac{1+2t+\cdots+(d_i-1)t^{d_i-2}}{1+t+\cdots+t^{d_i-1}}\Big)$$

$$= \frac{1}{|G|}\Big[\sum_{g\in H}\frac{\lambda(g)-1}{(1-\lambda(g)t)^2}+\sum_{g\notin H, g\neq 1}\frac{n(1-t)^{n-1}\det(1-gt)-(1-t)^n(\det(1-gt))'}{(\det(1-gt))^2}\Big] \quad (2)$$

Since $\det(1-gt)$ is not divisible by $(1-t)^{n-1}$ for $g \notin H, g \neq 1$ when t=1 we have

$$-\frac{1}{d_1\cdots d_n}\sum_{i=1}^{n}\frac{(1+2+\cdots+d_i-1)}{d_i} = -\frac{1}{|G|}\sum_{g\in H}\frac{1}{1-\lambda(g)} \quad (3)$$

Substituting $|G| = d_1 \cdots d_n$ into (3) so that

$$\sum_{i=1}^{n}\frac{d_i-1}{2} = \sum_{g\in H}\frac{1}{1-\lambda(g)} \quad (4)$$

The reflections of G fixing a hyperplane W and 1 form a cyclic subgroup G_W as shown in 2.1.3. Let $|G_W| = t$. Then $\lambda = \lambda(g)$ is a primitive t^{th} root of unity and

the elements of $H \cap G_W$ are reflections with $\lambda(g^k) = \lambda^k$ for $k = 1, \cdots, t-1$. Using the identity $\frac{1}{1-\lambda^i} + \frac{1}{1-\lambda^{t-i}} = 1$ for $i = 1, \cdots, t-1$, we may pair suitable terms to obtain

$$\sum_{h \in H \cap G_W} \frac{1}{1-\lambda(h)} = \frac{1}{2}(t-1) = \frac{1}{2}|H \cap G_W| \tag{5}$$

Since there are only a finite number of reflections generating G, H can be written as a finite disjoint union $H = \bigcup_{i=1}^{p} H \cap G_{W_i}$. Then using (4) and (5), we have

$$\begin{aligned} |H| = \sum_{j=1}^{p} |H \cap G_{W_i}| \qquad &= 2\sum_{j=1}^{p} \sum_{h \in H \cap G_{W_i}} \frac{1}{1-\lambda(h)} \\ = 2\sum_{g \in H} \frac{1}{1-\lambda(g)} \qquad &= \sum_{i=1}^{n} (d_i - 1) \end{aligned}$$

which proves the result. ∎

The following proposition gives a method to verify that a certain set of G-invariants generates S^G and hence is a set of basic invariants for G.

Proposition 5.3.3. *Any set of algebraically independent homogeneous G-invariants $g_1, \cdots, g_n$ with degrees $e_1, \cdots, e_n$ satisfying $\prod_{i=1}^{n} e_i = |G|$ is a set of basic invariants.*

Proof: Assume $e_1 \leq \cdots \leq e_n$. Let $f_1, \cdots, f_n$ be a set of basic invariants of degrees $d_1 \leq \cdots \leq d_n$. Since g_1 is a polynomial in $f_1, \cdots, f_n$ we find $e_1 \geq d_1$. We claim $e_i \geq d_i$ for all i. If not , there exists a minimal index k for which $e_k < d_k$. Then each of $g_1, \cdots, g_k$ is a polynomial in $f_1, \cdots, f_{k-1}$. But this contradicts the fact that $K(g_1, \cdots, g_k)$ has transcendence degree k over K. The claim follows.

Since $\prod_{i=1}^{n} d_i = |G| = \prod_{i=1}^{n} e_i$ by Proposition 5.3.1 we find $d_i = e_i$ for all i. Then $K(g_1, \cdots, g_d) = K(f_1, \cdots, f_d)$ for each d so that $K[g_1, \cdots, g_n] = K[f_1, \cdots, f_n] = S^G$. ∎

This proposition allows us to determine a set of basic invariants and the degrees of some finite reflection groups.

Example 5.3.4. [Spr] A set of basic invariants for $G(n,m,d)$ defined in Example 2.1.5(f) is given by

$$f_k = \sigma_k(x_1^m, \cdots, x_n^m)\ k = 1, \cdots, n-1 \qquad f_n = (x_1 \cdots x_n)^d$$

Note that $d_k = \deg f_k = mk$ for $k = 1, \cdots, n-1$ and $d_n = dn$ so that

$$\prod_{i=1}^{n} d_i = dm^{n-1}n! = |G(n,m,d)|$$

and

$$\sum_{i=1}^{n}(d_i - 1) = m\frac{n(n-1)}{2} + n(d-1)$$

is the number of reflections of $G(n,m,d)$. By a well known result (Lang V9) the n elementary symmetric polynomials in n variables are algebraically independent. So $f_1, \cdots, f_{n-1}, f_n^p$ where pd=n are algebraically independent and hence $f_1, \cdots, f_n$ are algebraically independent. We have found the degrees $d_1, \cdots, d_n$ for $G(n,m,d)$. By our observations in 2.1.5(f), this means that we have also found a set of basic invariants and hence the degrees for $\mathcal{S}_{n+1}$, $\mathcal{A}_n$, $\mathcal{B}_n$, $\mathcal{D}_n$ and $\mathcal{D}_2^m$. The results are summarized in the following table.

Table 5.3.1: Basic Invariants

W	$\{f_i\}$	Degrees
$\mathcal{S}_{n+1}$	$\sigma_k(x_1, \cdots, x_n), k = 1, \cdots, n$	$1, \cdots, n+1$
$\mathcal{A}_n$	$\sigma_k(x_1, \cdots, x_n), k = 1, \cdots, n$	$2, \cdots, n+1$
$\mathcal{B}_n$	$\sigma_k(x_1^2, \cdots, x_n^2), k = 1, \cdots, n$	$2, 4, \cdots, 2n$
$\mathcal{D}_n$	$\sigma_k(x_1^2, \cdots, x_n^2), k = 1, \cdots, n-1$	$2, 4, \cdots, 2(n-1), n$
	$x_1 \cdots x_n$	
$\mathcal{D}_2^m$	$x_1x_2, x_1^m + x_2^m$	$2, m$

Section 5.4: Converse to Chevalley's Theorem

The converse to Chevalley's Theorem shows that if the algebra of invariants S^G is a graded polynomial algebra, then G is a finite reflection group generated by the reflections it contains. To prove this theorem attributed to Shephard and Todd, we recall the Jacobian criterion for algebraic independence. We may apply this result to the symmetric algebra S under its canonical identification with $K[X_1, \ldots, X_n]$ where $\{x_1, \ldots, x_n\}$ is a basis for V.

Proposition 5.4.1. *A set $\{f_1, \ldots, f_n\} \subset K[X_1, \ldots, X_n]$ is algebraically independent if and only if the determinant $J(f_1, \ldots, f_n)$ of the Jacobian matrix is non-zero.*

Proof: Suppose $\{f_1, \ldots, f_n\}$ is an algebraically dependent set in $K[x_1, \ldots, x_n]$. Then we will show that $J(f_1, \ldots, f_n) = 0$ by showing that the Jacobian matrix is singular.

There exists $h \in K[y_1, \ldots, y_n]$ with $h(f_1, \ldots, f_n) = 0$. Without loss of generality, we may assume that h is homogeneous of minimal degree. Then for each $j = 1, \ldots, n$,

$$\sum_{i=1}^{n} \frac{\partial h}{\partial y_i}(f_1, \ldots, f_n)\frac{\partial f_i}{\partial x_j} = 0$$

In other words, Jv=0 where $J_{ij} = \left(\frac{\partial f_i}{\partial x_j}\right)$ is the Jacobian matrix and v is the vector with coordinates $v_i = \frac{\partial h}{\partial y_i}(f_1, \ldots, f_n)$. It suffices to show that $v_i \neq 0$ for some i. But there exists an i with $\frac{\partial h}{\partial y_i} \not\equiv 0$ as h is not a constant polynomial.Then $\frac{\partial h}{\partial y_i}$ is a non-zero homogeneous polynomial of strictly lower degree than h. By minimality of h, $v_i = \frac{\partial h}{\partial y_i}(f_1, \ldots, f_n) \neq 0$. Hence J is singular so that $J(f_1, \ldots, f_n) = 0$.

Conversely, suppose $\{f_1, \ldots, f_n\}$ is an algebraically independent set in $K[X_1, \ldots, X_n]$. Since $K(X_1, \ldots, X_n)$ has transcendence degree n over K, $\{x_i, f_1, \ldots, f_n\}$ is an algebraically dependent set for each i. Hence we have a homogeneous polynomial $h_i \in K[Y_0, Y_1, \ldots, Y_n]$ of minimal degree such that

$$h_i(x_i, f_1, \ldots, f_n) = 0 \; \forall i.$$

Then differentiating with respect to x_k for each k, we obtain

$$\frac{\partial h_i}{\partial y_0}(x_i, f_1, \ldots, f_n)\delta_{ik} + \sum_{j=1}^{n} \frac{\partial h_i}{\partial y_j}(x_i, f_1, \ldots, f_n)\frac{\partial f_j}{\partial x_k} = 0$$

Since $\{f_1, \ldots, f_n\}$ is an algebraically independent set in $K[x_1, \ldots, x_n]$, each h_i involves a positive power of y_0 so that $\frac{\partial h_i}{\partial y_0} \neq 0$ for all i. Again, since each $\frac{\partial h_i}{\partial y_0}$ is homogeneous of strictly lower degree than h_i, by the minimality of the h_i , we have $\frac{\partial h_i}{\partial y_0} \neq 0$. In terms of matrices we have:

$$\left(\frac{\partial h_i}{\partial y_j}(x_i, f_1, \ldots, f_n)\right)\left(\frac{\partial f_i}{\partial x_j}\right) = \left(-\frac{\partial h_i}{\partial y_0}(x_j, f_1, \ldots, f_n)\delta_{ij}\right)$$

Since the matrix on the right is a diagonal matrix with non-zero entries on the diagonal, its determinant is non-zero so that $J(f_1, \ldots, f_n) = \det\left(\frac{\partial f_i}{\partial x_j}\right) \neq 0$. ∎

Corollary 5.4.2. *Suppose $\{f_1, \ldots, f_n\}$ is an algebraically independent set of homogeneous elements of degrees $d_1, \ldots, d_n$. Then $J(f_1, \ldots, f_n)$ is homogeneous of degree $\sum_{i=1}^{n}(d_i - 1) = N$.*

Proof:

$$J(f_1, \ldots, f_n) = \sum_{\pi \in S_n} (-1)^{\operatorname{sgn}\pi} \frac{\partial f_1}{\partial x_{\pi(1)}} \cdots \frac{\partial f_n}{\partial x_{\pi(n)}}$$

By the proposition, $J(f_1, \ldots, f_n) \neq 0$. So there exist non-zero terms of the form

$$\frac{\partial f_1}{\partial x_{\pi(1)}} \cdots \frac{\partial f_n}{\partial x_{\pi(n)}}.$$

As each $\frac{\partial f_i}{\partial x_j}$ is homogeneous of degree $d_i - 1$, each non-zero term $\frac{\partial f_1}{\partial x_{\pi(1)}} \cdots \frac{\partial f_n}{\partial x_{\pi(n)}}$ is homogeneous of degree $\sum_{i=1}^{n}(d_i - 1)$. Hence the result follows. ∎

Now we are ready to prove the converse to Chevalley's Theorem.

Theorem 5.4.3. *(Shephard, Todd) [ST] Let V be an n-dimensional vector space over a field K of characteristic 0 and let G be a finite subgroup of $GL(V)$. Then the action of G can be naturally extended to the symmetric algebra S of V. Suppose the*

algebra of invariants S^G is a graded polynomial algebra generated by n algebraically independent homogeneous invariants $g_1, \ldots, g_n$. Then G is a finite reflection group generated by the reflections it contains.

Proof: We identify S with $K[X_1, \ldots, X_n]$ where $\{x_1, \ldots, x_n\}$ is a basis for V. Let H be the subgroup of G generated by all the reflections of G. Then by Chevalley's Theorem, S^H is a polynomial algebra generated by n algebraically independent homogeneous polynomials $f_1, \ldots, f_n$. Let $\deg(f_i) = d_i$ and $\deg(g_i) = e_i$ for all i. Since $S^G \subset S^H$, we may express each g_i as a polynomial h_i in the f_i's. Since g_i is homogeneous of degree e_i, we may assume that for each monomial $cf_1^{k_{i1}} \cdots f_n^{k_{in}}$ occurring in g_i, $\sum_{j=1}^n k_{ij} d_j = e_i$.

We wish to show $e_i \geq d_i$ for each i so that we may use the results on the sum and product of the degrees to get the result. Differentiating with respect to x_k we obtain

$$\frac{\partial g_i}{\partial x_k} = \sum_j \frac{\partial g_i}{\partial f_j} \frac{\partial f_j}{\partial x_k}$$

Since the Jacobian $J(g_1, \ldots, g_n)$ is non-zero, so is $\det\left(\frac{\partial g_i}{\partial f_j}\right)$. Then

$$\sum_{\pi \in S_n} (-1)^{\operatorname{sgn}\pi} \frac{\partial g_1}{\partial f_{\pi(1)}} \cdots \frac{\partial g_n}{\partial f_{\pi(n)}} \neq 0$$

There exists $\pi \in S_n$ with $\frac{\partial g_1}{\partial f_{\pi(1)}} \cdots \frac{\partial g_n}{\partial f_{\pi(n)}} \neq 0$. By rearranging the order of the f_i, assume $\frac{\partial g_1}{\partial f_1} \cdots \frac{\partial g_n}{\partial f_n} \neq 0$. Then $\frac{\partial g_i}{\partial f_i} \neq 0$ for all i. Hence $e_i \geq d_i$ for all i.

But then, since the number of reflections N in G and H is the same we have

$$\sum_i (d_i - 1) = N = \sum (e_i - 1)$$

so that $e_i = d_i$ for all i. Hence $|G| = \prod_{i=1}^n e_i = |H|$ implies that $H = G$. ∎

CHAPTER 6

Exponents of a Finite Real Reflection Group

In this chapter we will show that the eigenvalues of the elements of a certain special conjugacy class of the finite real orthogonal reflection group W completely determine the degrees of the basic invariants. We will assume that W is irreducible and essential. The exposition in this chapter is based on material from [Car] and [H2].

Definition 6.1.1. A **Coxeter element** of W is a product of the reflections s_{α_i}, where $\Delta = \{\alpha_i\}$ is a base for the root system Φ associated with W.

Theorem 6.1.2. *The Coxeter elements form a conjugacy class in W.*

Proof An element conjugate to a Coxeter element is again a Coxeter element since

$$w(s_{\alpha_1} \cdots s_{\alpha_n})w^{-1} = s_{w\alpha_1} \cdots s_{w\alpha_n}$$

and $w\Delta$ forms a base for Φ.

By the above calculation, we may see that it is sufficient to show that the Coxeter elements corresponding to a fixed base $\Delta = \{\alpha_i\}_{i=1}^n$ are conjugate since all the bases of Φ are W-conjugate. We may further reduce the problem by noting that a Coxeter element obtained by cyclic permutations of the factors in a given Coxeter element is conjugate to the original since

$$s_i s_{i+1} \cdots s_n s_1 \cdots s_{i-1} = s_i \cdots s_n (s_1 \cdots s_n) s_n \cdots s_i$$

Also interchanging adjacent commuting pairs of reflections s_i and s_j leave the Coxeter element unchanged. Hence we now need only show that all permutations of $1, \cdots, n$ can be performed by the following procedure. Write the indices 1 to n in a circle and then perform a sequence of interchanges of i and j where i and j

are adjacent on the circle but α_i and α_j are not connected on the Coxeter graph. Finally, find the resulting permutation by reading clockwise around the circle from a suitable point.

To prove the above statement we use induction on n. For n=1,2 the result is clear. Assume $n \geq 3$. Looking at the list of connected positive definite Coxeter graphs, we may assume that n is associated with a vertex of the Coxeter graph which is connected to one other vertex. By the inductive hypothesis, by removing n from the circle, we may perform a permutation of 1 through n-1 using only the operations described above. We must show that a permutation of $1, \cdots, n-1$ may be performed even when n is not removed. It suffices to prove that i and j corresponding to vertices α_i and α_j which are not joined in the graph may be interchanged even when n lies between them on the circle. Assume, without loss of generality, that α_n is not joined to α_i. Then the interchange of i and j may be performed by first interchanging i and n and then interchanging i and j. So it is possible to arrange $1, \cdots, n$ in the required order. Only n must be now moved into its correct position. But since α_n is joined to at most one of each pair α_i, α_j with $i, j < n$, it is possible to move n to the required position by successive interchanges clockwise or counterclockwise about the circle. ∎

Since all Coxeter elements are conjugate, they are similar in $GL(V)$ so that they have the same order and the same eigenvalues.

Definition 6.1.3.

(a) The **Coxeter number** h is the order of any Coxeter element of W.

(b) The **exponents** of the finite reflection group W are given by $m_1 \leq \cdots \leq m_n$ where $\{e^{\frac{2m_i i}{h}} | i = 1, \cdots, n\}$ is the list of eigenvalues of a Coxeter element.

Example 6.1.4. A_n: The reflection group associated with A_n is isomorphic to S_{n+1}. The simple reflection corresponding to the element $\epsilon_i - \epsilon_{i+1}$ of the base is $(i, i+1)$. Hence a Coxeter element is given by $w = (1, \cdots, n+1)$ so that the Coxeter

number is n+1. Since the characteristic polynomial of w is given by $\lambda^{n+1} - 1 = 0$, the eigenvalues of w are $e^{\frac{2\pi i}{n+1}}$ for $k = 1, \cdots, n$ so that the exponents of $\mathcal{A}_n$ are $1, \cdots, n$.

B_n: A Coxeter element of $\mathcal{B}_n$ is given by the transformation

$$\epsilon_i \to \epsilon_{i+1}, 1 \leq i \leq n$$

$$\epsilon_n \to -\epsilon_1$$

Hence the Coxeter number is 2n. Since the characteristic polynomial is $\lambda^n + 1 = 0$, the eigenvalues are $e^{\frac{2\pi i(1+2k)}{2n}}$ for $k = 0, 1, \cdots, n-1$ so that the exponents are $1, 3, \cdots, 2n-1$.

D_2^m: A Coxeter element is a rotation of order m. Here the characteristic polynomial is $\lambda^2 - 2\cos\frac{2\pi}{m} + 1 = 0$. The eigenvalues are $e^{\pm\frac{2\pi i}{m}}$ so that the exponents are 1 and m-1.

We wish to show that the degrees of the basic invariants are found by increasing each of the exponents by 1. Since the smallest degree is 2 as W is effective, we would expect that the smallest exponent is 1.

Lemma 6.1.5. *A Coxeter element of W has no eigenvalue equal to 1. The numbers $h - m_i$ are a permutation of the exponents of W and $\sum_{i=1}^{n} m_i = \frac{nh}{2}$.*

Proof: Suppose not. Then a Coxeter element $s_1 \cdots s_n$ fixes an element $v \in V$ so that $s_2 \cdots s_n v = s_1 v$. But this means $s_1 v - v = (v, \alpha_1)\alpha_1$ is also a linear combination of the roots $\alpha_i, i \geq 2$ of the base. Since these roots are linearly independent, we find that $(v, \alpha_1) = 0$ and $s_2 \cdots s_n v = v$. By a repetition of this argument we find that v is orthogonal to all the elements of the base. Since Δ spans V, then $v = 0$.

Since $w = s_1 \cdots s_n$ is a real linear transformation of V, its eigenvalues occur in complex conjugate pairs with associated exponents m_i and $h - m_i$. We know that 1 is not an eigenvalue of W. Assume -1 is an eigenvalue. Then $-1 = \zeta^{\frac{h}{2}}$ where $\zeta = e^{\frac{2\pi i}{h}}$. But the associated exponent $m_i = \frac{h}{2} = h - m_i$. Then the $h - m_i$ are a permutation of the exponents m_i so that $\sum_{i=1}^{n} m_i = \frac{nh}{2}$. ∎

We need a means by which to determine the eigenvalues of a Coxeter element of W. We will show that a Coxeter element may be embedded into a dihedral subgroup of W so that it acts faithfully as a rotation on a 2 dimensional subspace of V. The proofs that follow will assume that the dimension of V is greater than 1. Note that the only connected effective reflection group of dimension 1 is A_1. Here the connection between the exponents and degrees is clear.

Lemma 6.1.6. *A base Δ may be divided into two disjoint subsets such that each subset contains mutually orthogonal elements.*

Proof: Use induction on the number of roots of Δ. Remove a vertex on the Coxeter graph which is connected to at most one other vertex. The corresponding root is orthogonal to all but one other root of Δ. The remaining roots of Δ are divided into two subsets with mutually orthogonal elements. Put the remaining root into the appropriate subset. ∎

We may assume that $\Delta = \{\beta_1, \cdots, \beta_n\}$ is decomposed into two subsets $\{\beta_1, \cdots, \beta_k\}$ and $\{\beta_{k+1}, \cdots, \beta_n\}$ according to the lemma. Set $\alpha_i := \frac{\beta_i}{|\beta_i|}$.

Set $w_1 = s_1 \cdots s_k$ and $w_2 = s_{k+1} \cdots s_n$ where $s_i = s_{\alpha_i}$. w_1 and w_2 are elements of order 2 since they are products of commuting reflections. So $w = w_1 w_2$ is a Coxeter element and $\langle w_1, w_2 \rangle$ generates a dihedral subgroup of W. We now must find a suitable plane on which $w_1 w_2$ acts as a rotation.

Recall the symmetric positive definite matrix defined by $A_{ij} = (\alpha_i, \alpha_j)$. It has ones on the diagonal and non-positive off-diagonal entries. Since W is assumed to be connected, A is indecomposable: that is, it is impossible to divide the indices $\{1, \cdots, n\}$ into 2 non-empty complementary subsets I and J with $a_{ij} = 0$ when $i \in I, j \in J$.

We will need the following general theorem on real symmetric positive semi-definite indecomposable matrices.

Theorem 6.1.8. *(Perron - Frobenius) Let A be a real symmetric positive semi-definite indecomposable matrix with $a_{ij} \leq 0$ for $i \neq j$. Then the eigenvalues of A are real and non-negative. The smallest eigenvalue of A occurs with multiplicity 1 and has a corresponding eigenvector with strictly positive coefficients.*

Proof The eigenvalues are real since A is symmetric and non-negative since A is positive semi-definite. There exists an orthogonal matrix P such that $P^T AP$ is a diagonal matrix D. By using this fact it may be seen that the null space of A, V_0 is equal to the space $\{x \in \mathbf{R} | x^T Ax = 0\}$. We claim that $\dim V_0 \leq 1$.

Suppose $\dim V_0 > 0$. We will show that any non-zero vector x of $V - \{0\}$ has all coefficients non-zero. Let y be the vector with $y_i = |x_i|$. Then

$$0 \leq y^T Ay \leq x^T Ax = 0$$

since $a_{ij} \leq 0$ if $i \neq j$. This forces $y \in V_0$ and hence for all i,

$$\sum_{k=1}^{n} a_{ik} |x_k| = 0. \qquad (*)$$

Set

$$I = \{i | x_i = 0\}$$

$$J = \{i | x_i \neq 0\}$$

Then I and J are complementary subsets of $\{1, \cdots, n\}$ and J is non-empty. We use the hypothesis that A is indecomposable to show that I is empty. The sum (*) may be taken over $j \in J$. We find that each term in the sum is non-positive. If $i \in I$ and $j \in J$ then $a_{ij} = 0$. So we find I is empty and x has all coefficients non-zero. As this is true for any non-zero vector of V_0, then $\dim V_0 \leq 1$. If $\dim V_0 = 1$, then V_0 contains a vector with all coefficients positive.

Let μ be the minimal eigenvalue of A. By using the fact that $P^T AP = D$ we find that $A - \mu I$ also satisfies the hypothesis of the theorem. Since $A - \mu I$ is singular, its null-space is 1-dimensional and contains a vector with all coefficients positive. The theorem follows. ∎

Let $\{\hat{\alpha}_i\}_{i=1}^n$ be the dual basis of Δ with respect to the inner product, ie $(\alpha_i, \hat{\alpha}_j) = \delta_{ij}$. Then A is the change of basis matrix from the dual basis to Δ. That is, for all i,

$$\alpha_i = \sum_{j=1}^{n} (\alpha_i, \alpha_j)\hat{\alpha}_j.$$

Similarly, for all i,

$$\hat{\alpha}_i = \sum_{j=1}^{n} (\hat{\alpha}_i, \hat{\alpha}_j)\alpha_j.$$

This means that the symmetric matrix A^{-1} has ij entry $(\hat{\alpha}_i, \hat{\alpha}_j)$.

Apply the previous theorem to our matrix $A_{ij} = (\alpha_i, \alpha_j)$. If μ is the smallest eigenvalue of A then $\lambda = 1 - \mu$ is the largest eigenvalue of the symmetric $I - A$. Then the corresponding eigenvector, $u = (\zeta_i)_{i=1}^n$ can then be chosen with all coefficients positive.

We now define

$$a = \sum_{i=1}^{k} \zeta_i \hat{\alpha}_i \qquad b = \sum_{i=k+1}^{n} \zeta_i \hat{\alpha}_i.$$

We will show that W acts as a rotation on the plane spanned by a and b. The following lemma shows that a and b lie on a circle Γ in the plane and that the angle between a and b is determined by λ.

Lemma 6.1.9.

(i) (a,a)=(b,b)

(ii) The angle θ between a and b is given by $\cos\theta = \lambda$.

Proof A may be written as the block matrix

$$\begin{bmatrix} I_k & M \\ M^T & I_{n-k} \end{bmatrix}$$

where M is a $k \times (n-k)$ matrix and A^{-1} may be written as a block matrix with blocks corresponding to those of A

$$A^{-1} = \begin{bmatrix} B & C \\ C^T & D \end{bmatrix}.$$

Also, we write

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

where $u_1 \in \mathbf{R}^k, u_2 \in \mathbf{R}^{n-k}$.

$(I - A)u = \lambda u$ implies that

$$-Mu_2 = \lambda u_1 \qquad -M^T u_1 = \lambda u_2.$$

$A^{-1}A = I_n$ implies that

$$BM + C = 0 \qquad C^T + DM^T = 0.$$

Then

$$Cu_2 = -BMu_2 = \lambda Bu_1 \qquad C^T u_1 = -DM^T u_1 = \lambda Du_2 \qquad (*)$$

But

$$(a, a) = \sum_{i,j=1}^{k} \zeta_i \zeta_j (\hat{\alpha}_i, \hat{\alpha}_j) = u_1^T B u_1$$

and

$$(b, b) = \sum_{i,j=k+1}^{n} \zeta_i \zeta_j (\hat{\alpha}_i, \hat{\alpha}_j) = u_2^T D u_2.$$

So

$$u_1^T B u_1 = \lambda^{-1} u_1^T C u_2 = \lambda^{-1} (C^T u_1)^T u_2 = \lambda^{-1} (\lambda D u_2)^T u_2 = u_2^T D u_2$$

implies that $(a, a) = (b, b)$. We may find the angle between a and b using

$$|a||b| \cos\theta = (a, b) = \sum_{i=1}^{k} \sum_{j=k+1}^{n} \zeta_i \zeta_j (\hat{\alpha}_i, \hat{\alpha}_j)$$

$$= u_1^T C u_2 = u_1^T \lambda B u_1 = \lambda (a, a)$$

Then $(a, a) = (b, b)$ implies that $\cos\theta = \lambda$. ∎

Proposition 6.1.10. *Let a and b be defined as above and let Γ be the circle with centre the origin passing through a and b. Then the dihedral subgroup $\langle w_1, w_2 \rangle$ of*

W acts on Γ. In particular, w_1 is the reflection in the line $\mathbf{R}b$, w_2 is the reflection in the line $\mathbf{R}a$, and w is a rotation around Γ through the angle 2θ where $\cos\theta = \lambda$.

Proof: Note that $(a, \lambda a - b) = (a,a)\lambda - (a,b) = 0$ by the lemma. To show that w_2 is a reflection in the line $\mathbf{R}a$ is equivalent to showing that w_2 is a reflection in $\lambda a - b$. But for $i = 1, \cdots, k$,

$$(\hat{\alpha}_i, \lambda a - b) = \sum_{j=1}^{k} \lambda \zeta_j (\hat{\alpha}_i, \hat{\alpha}_j) - \sum_{j=k+1}^{n} \zeta_j (\hat{\alpha}_i, \hat{\alpha}_j) = 0$$

since $\lambda B u_1 - C u_2 = 0$ by (*) in the Lemma. Hence $\lambda a - b$ is a linear combination of $\alpha_{k+1}, \cdots, \alpha_n$ so that $w_2(\lambda a - b) = -(\lambda a - b)$. Since w_2 also fixes a, we see that w_2 stabilizes the plane $\mathbf{R}a \oplus \mathbf{R}b$ and acts as a reflection in $\mathbf{R}a$. Similarly by the lemma $(b, a - \lambda b) = 0$ so that it suffices to show that w_1 is a reflection in $a - \lambda b$. Again for $i = 1, \cdots, n, (\hat{\alpha}_i, a - \lambda b) = 0$ as $C^T u_1 - \lambda D u_2 = 0$ by the lemma. So $w_1(a - \lambda b) = -(a - \lambda b)$ and $w_1(b) = b$ imply that w_1 acts on Γ as a reflection in $\mathbf{R}b$.

Then $w = w_1 w_2$ is a rotation in 2θ in the plane of Γ where θ is the angle between a and b. ∎

Lemma 6.1.11. *Let a, b be chosen as above with respect to the largest eigenvalue λ of $I - A$ and its corresponding eigenvector u with $\zeta_i > 0$. Then any point c lying on Γ strictly between a and b lies in the fundamental chamber.*

Proof: Since $a = \sum_{i=1}^{k} \zeta_i \hat{\alpha}_i$ and $b = \sum_{i=k+1}^{n} \zeta_i \hat{\alpha}_i$, we find that c lying strictly between a and b on Γ must take the form $c = \sum_{i=1}^{n} \eta_i \hat{\alpha}_i$ where $\eta_i > 0$. But then $(\alpha_i, c) = \eta_i > 0$ for all i so that $c \in C(\Delta)$. ∎

We have already seen that the sum of the exponents is $\frac{nh}{2}$ and the sum of the degrees is N, the number of positive roots in Φ. Suspecting the connection between the exponents and the degrees, the statement of the following theorem which gives a formula for the Coxeter number is predictable.

Theorem 6.1.12. *The Coxeter number h is $\frac{2N}{n}$.*

Proof: Let w be a Coxeter element of order h and Γ be the circle defined in Proposition 6.1.10. Then w acts as a rotation on Γ. In fact, $\langle w \rangle$ acts faithfully on Γ. Suppose w acts as the identity on Γ. Then by the last lemma, it fixes an element of $C(\Delta)$ so that by Theorem 3.2.8(a), $w = 1$. This implies that w has order h on Γ.

We now show that the angle between a and b is $\frac{\pi}{h}$. Let $c = w_2 b$ be the reflection of b in $\mathbf{R}a$. The angle between b and c is 2θ. Since w acts as a rotation in 2θ on Γ, there exists $i > 0$ with $w^i a$ lying in the arc bc. Let i be the least such positive integer. If $w^i a$ lies in the arc ba, then a and $w^i a$ lie in $\overline{C}(\Delta)$ so that by 3.2.8(b) $w^i a = a$. Otherwise $w^i a$ lies in the arc ac so that $w^i b$ lies in the arc ba. We find that $w^i b = b$. Since $\langle w \rangle$ acts faithfully on Γ, we find $i = h$ and $\theta = \frac{\pi}{h}$.

Next we show that a lies in exactly n-k hyperplanes. Since $a = \sum_{i=1}^{k} \zeta_i \hat{\alpha}_i$, a is orthogonal to $\pm\alpha_i$ for $i = k+1, \cdots, n$.

Suppose $d = \sum_{i=1}^{n} \eta_i \alpha_i$ is a root orthogonal to a. Then $\sum_{i=1}^{k} \zeta_i \eta_i = 0$. Since η_i must be of the same sign and $\zeta_i > 0$, we find that $\eta_i = 0$ for all $i = 1, \cdots, k$. Then $d = \sum_{i=k+1}^{n} \eta_i \alpha_i$. But $J = \{\alpha_i\}_{i=k+1}^{n}$ forms a base for the root system $\Phi_J = \text{span}(J) \cap \Phi$. Since $\{\alpha_i\}_{i=k+1}^{n}$ are mutually orthogonal, we find that $\Phi_J = \{\pm\alpha_i\}$. Thus a lies in exactly n-k hyperplanes. Since $wH_\alpha = H_{w\alpha}$ for any $w \in W$, every translate $w^i a$ lies in exactly n-k hyperplanes. Similarly b and its translate $w^i b$ lie in exactly k hyperplanes.

Lastly we show that a reflecting hyperplane must intersect Γ in a translate $w^i a$ or $w^j b$. If not, then some translate of the hyperplane intersects Γ at a point strictly between a and b. By Lemma 6.1.11, this point must lie in the fundamental chamber. By contradiction, the intersection points between reflecting hyperplanes are translates of a and b. Since there are h translates of a and of b respectively, the number of reflecting hyperplanes is $\frac{1}{2}[(n-k)h + kh] = \frac{1}{2}nh$ because each hyperplane intersects Γ in two points. But then $N = \frac{1}{2}nh$ so that $h = \frac{2N}{n}$ ∎

We find that the sums of degrees and exponents do agree with the hypothesis.

Corollary 6.1.13. $\sum_{i=1}^{n} m_i = N = \sum_{i=1}^{n}(d_i - 1)$.

Corollary 6.1.14. *If the exponents of W are $m_1 \leq \cdots \leq m_n$ then $m_1 = 1$ and $m_n = h - 1$.*

Proof: Eigenvalues of Coxeter elements are of the form $e^{\frac{2\pi i m}{h}}$. $e^{\frac{2\pi i}{h}}$ occurs as an eigenvalue as w is a rotation of $\frac{2\pi}{h}$ in the plane of Γ. Since 1 does not occur as an eigenvalue we find that $m_1 = 1$. But complex eigenvalues of real tranformations occur in complex conjugate pairs. Hence $m_n = h - 1$. ∎

Table 6.1.1: Coxeter Number

W	$\mathcal{A}_n$	$\mathcal{B}_n$	$\mathcal{D}_n$	$\mathcal{E}_6$	$\mathcal{E}_7$	$\mathcal{E}_8$	$\mathcal{F}_4$	$\mathcal{G}_2$	$\mathcal{H}_3$	$\mathcal{H}_4$	$\mathcal{D}_2^m$
$\|W\|$	$(n+1)!$	$2^n n!$	$2^{n-1}n!$	$2^7 3^4 5$	$2^{10}3^4 5 \cdot 7$	$2^{14}3^5 5^2 7$	$2^7 3^2$	12	$2^3 3 \cdot 5$	$2^6 3^2 5^2$	$2m$
N	$\frac{n(n+1)}{2}$	n^2	n(n-1)	36	63	120	24	6	15	60	m
h	n+1	2n	2(n-1)	12	18	30	12	6	10	30	m

Example 6.1.15. From the last corollary, we may determine the exponents for all irreducible reflection groups of rank ≤ 3. For instance the exponents of $\mathcal{G}_2$ are 1 and 5 and for $\mathcal{D}_2^m$ they are 1 and m-1. For $\mathcal{H}_3$, we find the exponents 1 and 9. The remaining exponent must be 15 - 10 = 5 by 6.1.5.

Corollary 6.1.16. *A Coxeter element w has an eigenvector v with eigenvalue $e^{\frac{2\pi i}{h}}$. Then v does not lie in any hyperplane $H_\alpha, \alpha \in \Phi$.*

Proof: Let v be an eigenvector with eigenvalue $e^{\frac{2\pi i}{h}}$ lying in the complexified plane of Γ, $P_{\mathbf{C}} = \mathbf{C}a \oplus \mathbf{C}b$. Since w acts as a rotation, v is not real so that $v \neq \overline{v}$. Suppose $(v, \alpha) = 0$ for some $\alpha \in \Phi$. Then $(\overline{v}, \alpha) = 0$. But v and $\overline{v}$ span $P_{\mathbf{C}}$ so that

we find α is orthogonal to $P_{\mathbf{C}}$. But the plane $P_{\mathbf{C}}$ contains elements of $C(\Delta)$. By contradiction, v cannot lie in H_α for any $\alpha \in \Phi$. ∎

Finally we can relate the exponents to the degrees of the basic invariants.

Theorem 6.1.17. *(Coleman) If $m_1, \cdots, m_n$ are the exponents of W then the degrees of W are $m_1 + 1, \cdots, m_n + 1$.*

Proof: The eigenvalues of a Coxeter element are of the form ζ^m where $0 < m < h$ and $\zeta = e^{\frac{2\pi i}{h}}$. Take a base for the complexification $V_{\mathbf{C}}$ of V consisting of eigenvectors $y_1, \cdots, y_n$ where

$$w(y_i) = \zeta^{m_i} y_i.$$

Let $f_1, \cdots, f_n$ be a set of basic invariants in the complexification $S_{\mathbf{C}}$ of S of degrees $d_1, \cdots, d_n$. Each f_i may be identified with a polynomial in the y_i. The Jacobian J of $f_1, \cdots, f_n$ may be factored as $J = c\prod_{\alpha \in \Phi^+} \alpha$ where $c \in \mathbf{C}^*$ as will be shown in 7.1.3. By Corollary 6.1.16, $(\alpha, y_1) \neq 0$ for all $\alpha \in \Phi^+$. This means that $\frac{\partial \alpha}{\partial y_1} \neq 0$ and hence $\frac{\partial J}{\partial y_1} \neq 0$ so that J is not zero at $(1, 0, \cdots, 0)$. Hence at least one of the $n!$ products in the determinant is not 0 at this point. By suitable reordering we may assume $\frac{\partial f_i}{\partial y_i} \neq 0$ at $(1, 0, \cdots, 0)$for all i. Then

$$\frac{\partial f_i}{\partial y_i} = a_i y_1^{d_i - 1} + \text{terms involving } y_2, \cdots, y_n \text{ with } a_i \neq 0.$$

so that

$$f_i = a_i y_1^{d_i - 1} y_i + \text{different monomials.}$$

Applying w, we obtain

$$w(f_i) = f_i = a_i \zeta^{d_i - 1 + m_i} y_1^{d_i - 1} y_i + \text{different monomials.}$$

We find that $\zeta^{d_i - 1 + m_i} = 1$ for all i so that $d_i - 1 \equiv -m_i \equiv h - m_i \pmod{h}$. But $0 < m_i < h$ and $\sum_{i=1}^n (d_i - 1) = \sum_{i=1}^n (h - m_i)$ implies that $d_i - 1 = h - m_i$. Since the numbers $h - m_i$ are just a permutation of the exponents, we have our result. ∎

Example 6.1.18. Since we have already determined the degrees of the reflection groups corresponding to A_n, B_n, D_n, we also know their exponents from this theorem. We may also determine the exponents of the group of type H_4 with $N = 60, h = 30$ and $|W| = 14400$. By 6.1.14, we find 1 and 29 are two exponents. The other two exponents a,b have sum 30 and (a+1)(b+1)=240. It is easy to see that the exponents are 1,11,19,29.

In the special case of crystallographic finite reflection groups or Weyl groups, we may prove an additional result on the exponents.

Proposition 6.1.19. *Let W be an irreducible Weyl group with Coxeter number h. Then each $1 \leq m \leq h-1$ which is relatively prime to h is an exponent.*

Proof: The automorphism induced by a Coxeter element w may be represented by a matrix with respect to a base Δ for W. Since W is crystallographic the matrix will have integer entries. Hence the characteristic polynomial lies in $\mathbf{Z}[X]$. Since the primitive hth root of unity $\zeta = e^{\frac{2\pi i}{h}}$ occurs as an eigenvalue of w, the associated irreducible cyclotomic polynomial $\Phi(x)$ must divide the characteristic polynomial of w. But $\Phi(x)$ has as roots precisely the primitive hth roots of unity which in turn are ζ^m with m relatively prime to h and $1 \leq m \leq h-1$. ∎

Example 6.1.20. [H2] We may now determine the exponents (and hence degrees) of the remainder of the irreducible finite reflection groups. For $\mathcal{F}_4$, the exponents are 1,5,7,11 since these are the only numbers between 1 and 11 which are relatively prime to 12. Similarly, for $\mathcal{E}_8$ with $h = 30$, the exponents are 1,7,11,13,17,19,23,29. For $\mathcal{E}_7$ with $h = 18$ we can only find 6 exponents this way: 1,5,7,11,13,17. The remaining exponent satisfies $m = h - m$ so that $m = \frac{h}{2} = 9$. Lastly for $\mathcal{E}_6$ with $h = 12$, we find that 1,5,7 and 11 are exponents. We then use the fact that the sum of exponents is N=36 and the product of degrees is $|W| = 2^7 3^4 5$ to find the remaining exponents 4 and 8. We have found the exponents of the irreducible real

finite reflection groups.

Table 6.1.2: Exponents

Group	Exponents
$\mathcal{A}_n$	$1, 2, \cdots, n$
$\mathcal{B}_n$	$1, 3, \cdots, 2n-1$
$\mathcal{D}_n$	$1, 3, \cdots, 2n-3, n-1$
$\mathcal{E}_6$	$1, 4, 5, 7, 8, 11$
$\mathcal{E}_7$	$1, 5, 7, 9, 11, 13, 17$
$\mathcal{E}_8$	$1, 7, 11, 13, 17, 19, 23, 29$
$\mathcal{F}_4$	$1, 5, 7, 11$
$\mathcal{G}_2$	$1, 5$
$\mathcal{H}_3$	$1, 5, 9$
$\mathcal{H}_4$	$1, 11, 19, 29$
$\mathcal{D}_2^m$	$1, m-1$

CHAPTER 7

Solomon's Theorem and Its Generalizations

The proof of Solomon's result requires some knowledge of the invariant differential forms of a finite reflection group. We will obtain this information from the structure of S^G, the K-algebra of polynomial invariants of G. The material in this chapter is based on [Slm] and [PW].

We have already shown how G acts as a group of automorphisms of the symmetric algebra $S = S(V)$ and of the exterior algebra $\Lambda = \Lambda(V)$. It should now be clear how to extend the action of G on V to the graded K-algebra $S \otimes_K \Lambda$. Using the universal property of tensor products we find that for each $g \in G$, there exists a unique homomorphism $\rho(g)$ on $S \otimes \Lambda$ such that

$$\rho(g)(s \otimes e) = S(g)(s) \otimes \Lambda(g)(e)$$

for all $s \in S$ and $e \in \Lambda$. The uniqueness of the homomorphism above can be used to show that $\rho(1) = 1_{S \otimes \Lambda}$ and $\rho(gh) = \rho(g) \circ \rho(h)$ for all $g, h \in G$ so that in fact $[\rho(g)]^{-1} = \rho(g^{-1})$ and $\rho(g)$ is a graded K-algebra automorphism of $S \otimes \Lambda$. For simplicity, we denote the automorphism induced by $\gamma \in G$ on $S \otimes \Lambda$ by $u \mapsto \gamma(u)$. The graded K-algebra $(S \otimes \Lambda)^G$ of invariants under the action of G on $S \otimes \Lambda$ is called the K-algebra of invariant differential forms of G.

We next define a derivation on $S \otimes \Lambda$ which may be used to find a convenient expression for elements of $S \otimes \Lambda^p$. Let $\{x_1, \ldots, x_n\}$ be a basis for V. We identify S with $K[X_1, \ldots, X_n]$ to define the K-linear map

$$d : S \otimes \Lambda \to S \otimes \Lambda$$

$$d : (s \otimes x_{i_1} \wedge \cdots \wedge x_{i_p}) = \sum_{j=1}^{n} \frac{\partial s}{\partial x_j} \otimes x_j \wedge x_{i_1} \wedge \cdots \wedge x_{i_p}$$

Proposition 7.1.1. *[Slm] d is a derivation on $S \otimes \Lambda$ which commutes with the action of G on $S \otimes \Lambda$. In particular, $d : (S \otimes \Lambda)^G \to (S \otimes \Lambda)^G$.*

Proof: Since $d(s \otimes e) = d(s \otimes 1)(1 \otimes e)$ for all $s \in S, e \in \Lambda$ we need only verify $d(st \otimes 1) = d(s \otimes 1)(t \otimes 1) + (s \otimes 1)d(t \otimes 1)$ for all $s, t \in S$ to show that d is a derivation. This follows easily from the product rule for partial derivatives and linearity.

To show $d\gamma(s \otimes 1) = \gamma d(s \otimes 1)$ for all $s \in S, \gamma \in G$, we need only consider homogeneous $s \in S$. Proceeding by induction on the degree of $s \in S$ we note for $\deg s = 0$, $d\gamma(s \otimes 1) = 0 = \gamma(d(s \otimes 1))$.

For $\deg s = 1$, it suffices to show $d\gamma(x_j \otimes 1) = \gamma d(x_j \otimes 1)$ for all j. Let $\gamma x_j = \sum_{k=1}^{n} \gamma_{jk} x_k$. Then

$$d\gamma(x_j \otimes 1) = \sum_{k=1}^{n} \gamma_{jk} d(x_k \otimes 1) = \sum_{k=1}^{n} \gamma_{jk}(1 \otimes x_k)$$

$$= 1 \otimes \gamma(x_j) = \gamma(d(x_j \otimes 1))$$

Assume $s \in S$ is homogeneous of positive degree. Then $s = \sum_{i=1}^{n} s_i x_i$ for some $s_i \in S$. Using linearity, the derivation property and the inductive hypothesis we find

$$d\gamma(s \otimes 1) = d\gamma(\sum_{i=1}^{n} s_i x_i \otimes 1) = \sum_{i=1}^{n} d\gamma(s_i x_i \otimes 1)$$

$$= \sum_{i=1}^{n} [d(\gamma(s_i) \otimes 1)(\gamma(x_i) \otimes 1) + (\gamma(s_i) \otimes 1)d(\gamma(x_i) \otimes 1)]$$

$$= \sum_{i=1}^{n} [\gamma(d(s_i \otimes 1))\gamma(x_i \otimes 1) + \gamma(s_i \otimes 1)\gamma(d(x_i \otimes 1))]$$

$$= \gamma(d(\sum_{i=1}^{n} s_i x_i \otimes 1)) = \gamma(d(s \otimes 1))$$

Thus we have $d\gamma(s \otimes 1) = \gamma d(s \otimes 1)$ for all $s \in S$. Since $d(s \otimes e) = d(s \otimes 1)(1 \otimes e)$ for all $s \in S, e \in \Lambda$, we see that $d\gamma(s \otimes e) = \gamma d(s \otimes e)$. By linearity, d commutes with the action of G on $S \otimes \Lambda$.

Suppose $v \in (S \otimes \Lambda)^G$. Then for all $\gamma \in G, \gamma(dv) = d(\gamma v) = dv$.It follows that d maps $(S \otimes \Lambda)^G$ to $(S \otimes \Lambda)^G$. ∎

Identifying S with $S \otimes K$ we may write $dx_i = d(x_i \otimes 1) = 1 \otimes x_i$. Thus any element of $S \otimes \Lambda^p$ may be written uniquely as

$$\sum_{i_1 < \cdots < i_p} s_{i_1 \cdots i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p}$$

since $\{x_{i_1} \wedge \cdots \wedge x_{i_p} | i_1 < \cdots < i_p\}$ forms a basis of Λ over K.

For the last time, we extend the action of G over a new space. Let L be the quotient field of S. Recall that the action of G on L was defined by $g \cdot \left(\frac{s}{t}\right) = \frac{g \cdot s}{g \cdot t}$ for $g \in G, s \in S$ and $t \in S - \{0\}$. We wish to extend the action of G to $L \otimes_K \Lambda$. Analogous to our discussion of G acting on $S \otimes \Lambda$ we may show that for $g \in G$ there exists a unique graded L-algebra automorphism $\rho(g)$ of $L \otimes \Lambda$ with

$$\rho(g)(l \otimes e) = S(g)(l) \otimes \Lambda(g)(e)$$

for all $l \in L, e \in \Lambda$. We again denote the automorphism induced by $\gamma \in G$ on $L \otimes_K \Lambda$ by $y \mapsto \gamma(y)$. Note that $L \otimes_K \Lambda$ is a vector space over L.

We may now state the theorem which characterizes the K-algebra of invariant differential forms of a finite reflection group G.

Theorem 7.1.2. *[Slm] Let G be a finite reflection group in a K-vector space V. Let S be the symmetric algebra of V and identify S with $K[X_1, \ldots, X_n]$ where $\{x_1, \ldots, x_n\}$ is a basis of V. Let $\{f_1, \ldots, f_n\}$ be a set of homogeneous algebraically independent generators of the graded K-algebra S^G of polynomial invariants. Then the K-algebra $(S \otimes \Lambda)^G$ of invariant differential forms is an exterior algebra generated over S^G by the unit element and the differentials $df_1, \ldots, df_n$ of a set of basic invariants $f_1, \ldots, f_n$. In particular, every invariant differential p-form may be written uniquely as*

$$\sum_{i_1 < \cdots < i_p} a_{i_1 \cdots i_p} df_{i_1} \wedge \cdots \wedge df_{i_p}$$

where $a_{i_1 \cdots i_p} \in S^G$.

An element $s \in S$ is called **anti-invariant** if

$$\gamma(s) = (\det(\gamma))^{-1} s \ \forall \gamma \in G.$$

In order to prove this theorem we need a lemma which describes the anti-invariants of S under the action of G.

For this lemma we let H be the set of all reflections of G. Let $\gamma \in H$ be a reflection in $a_\gamma \in V$.

Lemma 7.1.3. *[Slm] The Jacobian j of a set of basic invariants $f_1, \ldots, f_n$ is an anti-invariant element of S under G which may be factored as $c\prod_{\gamma \in H} a_\gamma$ with $c \in K^*$. Every anti-invariant element of S may be factored as rj where $r \in S^G$.*

Proof: We first show that an anti-invariant element z of S under G is divisible by $D = \prod_{\gamma \in H} a_\gamma$ in S.

We showed in Corollary 2.1.4 that the reflections in a of G and 1 form a cyclic subgroup G_a of G. By Maschke's theorem, $V = Ka \oplus W$ where W is a hyperplane which is pointwise fixed by elements of G_a. Let $\{x_1, \ldots, x_n\}$ be a basis of V with $x_1 = a$ and $x_i \in W$ for all $i \neq 1$. Then z may be identified with a polynomial $p(x_1, \ldots, x_n) \in K[X_1, \ldots, X_n]$. Let $G_a = \langle \gamma \rangle$ and $\theta = \det(\gamma)$. Then $\gamma(z) = p(\theta x_1, x_2, \cdots, x_n) = \theta^{-1} p(x_1, \cdots, x_n)$. Differentiate this expression q times with respect to x_1 and set $x_1 = 0$ to obtain

$$\theta^q \frac{\partial^q p}{\partial x_1^q}(0, x_2, \cdots, x_n) = \theta^{-1} \frac{\partial^q p}{\partial x_1^q}(0, x_2, \cdots, x_n).$$

Since θ is a t^{th} root of unity ,

$$\frac{\partial^q p}{\partial x_1^q}(0, x_2, \cdots, x_n) = 0 \ \text{ for } q = 0, \cdots, t-2.$$

We claim a^{t-1} divides z. Let $a_2, \cdots, a_n$ be fixed values for $x_2, \cdots, x_n$ and let $g(x_1) = p(x_1, a_2, \cdots, a_n)$. The coefficient of x_1^q is $\frac{d^q g}{dx_1^q}(0) = 0$ for each q by the

Maclaurin expansion. So $x_1^{i-1} | g(x_1)$. But the choice of $a_2, \cdots, a_n$ was arbitrary so that $x_1^{i-1} = a^{i-1}$ divides z. Since the reflections of G are precisely the non-trivial elements of G_a for some unit vector $a \in V$ and distinct unit vectors of V are relatively prime in the factorial ring S , we find that $\prod_{\gamma \in H} a_\gamma$ divides z.

We now show that j is an anti-invariant element of S. Let $\varphi : V \to V$ be a K - linear map with $\varphi x_i = \sum_{j=1}^n \varphi_{ij} x_j$. Extend φ to a K - algebra homomorphism of $\Lambda^n(V)$. Then

$$\begin{aligned}
\varphi(dx_1 \wedge \cdots \wedge dx_n) &= \varphi(dx_1) \wedge \cdots \wedge \varphi(dx_n) \\
&= 1 \otimes (\sum_{j=1}^n \varphi_{1j} x_j \wedge \cdots \wedge \sum_{k=1}^n \varphi_{nk} x_k) \\
&= 1 \otimes (\sum_{\pi \in S_n} (-1)^{sgn\pi} \varphi_{1\pi(1)} \cdots \varphi_{n\pi(n)} x_1 \wedge \cdots \wedge x_n) \\
&= \det(\varphi) dx_1 \wedge \cdots \wedge dx_n
\end{aligned}$$

Then since $j dx_i = df_i$ for all i where j is the Jacobian transformation we find that

$$df_1 \wedge \cdots \wedge df_n = j(dx_1 \wedge \cdots \wedge dx_n)$$

Since $\gamma(dx_1 \wedge \cdots dx_n) = \det(\gamma)(dx_1 \wedge \cdots \wedge dx_n)$ for $\gamma \in G$ and the action of G commutes with d we have

$$\begin{aligned}
j(dx_1 \wedge \cdots \wedge dx_n) &= df_1 \wedge \cdots \wedge df_n \\
&= d(\gamma f_1) \wedge \cdots \wedge d(\gamma f_n) \\
&= \gamma(df_1 \wedge \cdots \wedge df_n) \\
&= \det(\gamma)\gamma(j) dx_1 \wedge \cdots \wedge dx_n
\end{aligned}$$

Hence $\gamma(j) = (\det\gamma)^{-1} j$ for all $\gamma \in G$ so that j is anti-invariant.

From our discussion above, we find that j is divisible by $\prod_{\gamma \in H} a_\gamma$, an element of degree $|H| = \sum_{i=1}^n (d_i - 1)$ where $d_i = \deg(f_i)$. But the Jacobian j is non-zero of the same degree. Hence $j = c \prod_{\gamma \in H} a_\gamma$ for some $c \in K^*$.

Let z be an arbitrary anti-invariant element of S. We see that $z = fj$ for some $f \in S$. Let $\gamma \in G$. In the quotient field L, we find

$$\gamma(f) = \frac{\gamma(z)}{\gamma(j)} = \frac{(\det\gamma)^{-1}z}{(\det\gamma)^{-1}j} = f$$

so that $f \in S^G$. Conversely, every element of the form fj for $f \in S^G$ is an anti-invariant element of S since

$$\gamma(fj) = \gamma(f)\gamma(j) = (\det\gamma)^{-1}fj \text{ for } \gamma \in G. \blacksquare$$

Proof of the Theorem:

To prove the theorem, we will show that $\{df_{i_1} \wedge \cdots \wedge df_{i_p} | i_1 < \cdots < i_p\}$ forms a basis for the vector space $L \otimes_K \Lambda^p$ over L. We will then use an averaging argument and the lemma to show that the coefficients of the basis elements for an invariant differential p-form must lie in S^G.

Since $L \otimes_K \Lambda^p$ has as basis $\{dx_{i_1} \wedge \cdots \wedge dx_{i_p} | i_1 < \cdots < i_p\}$ over L, its dimension is $\binom{n}{p}$. Hence we need only show that $\{df_{i_1} \wedge \cdots \wedge df_{i_p} | i_1 < \cdots < i_p\}$ is a linearly independent set over L.

Let $\sum_{i_1<\cdots<i_p} k_{i_1\cdots i_p} df_{i_1} \wedge \cdots \wedge df_{i_p} = 0$ with $k_{i_1\cdots i_p} \in L$. Fix a set of indices $i_1, \cdots, i_p$ and let $i_{p+1}, \cdots, i_n$ be the complementary subset in $1, \cdots, n$. Multiplying by $df_{i_1} \wedge \cdots \wedge df_{i_n}$ we find that $k_{i_1\cdots i_p} df_1 \wedge \cdots \wedge df_n = 0$. But then $k_{i_1\cdots i_p} j dx_1 \wedge \cdots \wedge dx_n = 0$. Since $j \neq 0$, we find $k_{i_1\cdots i_p} = 0$. Hence $\{df_{i_1} \wedge \cdots \wedge df_{i_p} | i_1 < \cdots < i_p\}$ forms a basis of the vector space $L \otimes \Lambda^p$ over L.

Let ω be an invariant differential p-form. Then

$$\omega = \sum_{i_1<\cdots<i_p} r_{i_1\cdots i_p} df_{i_1} \wedge \cdots \wedge df_{i_p}$$

with $r_{i_1\cdots i_p} \in L$. We wish to show that $r_{i_1\cdots i_p} \in S^G$. Apply the averaging operator α of the group G to ω. Since α projects the G-module $S \otimes \Lambda$ onto $(S \otimes \Lambda)^G$,

$$\omega = \alpha(\omega) = \sum_{i_1<\cdots<i_p} \frac{1}{|G|} \sum_{\gamma \in G} \gamma(r_{i_1\cdots i_p})\gamma(df_{i_1} \wedge \cdots \wedge df_{i_p})$$

$$= \sum_{i_1<\cdots i_p} \frac{s_{i_1\cdots i_p}}{t_{i_1\cdots i_p}} df_{i_1} \wedge \cdots \wedge df_{i_p}$$

where $\frac{s_{i_1\cdots i_p}}{t_{i_1\cdots i_p}} \in L^G$. Since ω is an element of $S \otimes \Lambda^p$, it may be also expressed in terms of the basis $\{dx_{i_1} \wedge \cdots \wedge dx_{i_p} | i_1 < \cdots < i_p\}$. Hence

$$\sum_{i_1<\cdots<i_p} u_{i_1\cdots i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p} = \sum_{i_1<\cdots<i_p} \frac{s_{i_1\cdots i_p}}{t_{i_1\cdots i_p}} df_{i_1} \wedge \cdots \wedge df_{i_p}$$

where $u_{i_1\cdots i_p} \in S$. Again we fix a set of indices $i_1, \cdots, i_p$ and multiply the equation above by $df_{i_{p+1}} \wedge \cdots \wedge df_{i_n} \in S \otimes \Lambda^{n-p}$ to obtain

$$w_{i_1\cdots i_p} dx_1 \wedge \cdots \wedge dx_n = \frac{s_{i_1\cdots i_p}}{t_{i_1\cdots i_p}} j dx_1 \wedge \cdots \wedge x_n$$

for some $w_{i_1\cdots i_p} \in S$. Then

$$\frac{w_{i_1\cdots i_p}}{j} = \frac{s_{i_1\cdots i_p}}{t_{i_1\cdots i_p}} \in L^G$$

so that for all $\gamma \in G$

$$\frac{w_{i_1\cdots i_p}}{j} = \frac{\gamma(w_{i_1\cdots i_p})}{\gamma(j)} = \frac{\gamma(w_{i_1\cdots i_p})}{(\det\gamma)^{-1} j}$$

which implies that $w_{i_1\cdots i_p}$ is an anti-invariant element of S. By the lemma,

$$v_{i_1\cdots i_p} = \frac{w_{i_1\cdots i_p}}{j} \in S^G.$$

Hence

$$\omega = \sum_{i_1<\cdots i_p} v_{i_1\cdots i_p} df_{i_1} \wedge \cdots \wedge df_{i_p}$$

where $v_{i_1\cdots i_p} \in S^G$. This proves the theorem. ∎

In fact, more generally, we have proved that for any finite subgroup G of $\mathrm{GL}(V)$, $(L \otimes \Lambda)^G$ is an exterior algebra over the field L^G of invariant rational functions generated over L^G by the unit element and the differentials of any n algebraically independent elements of S^G.

By Theorem 7.1.2, $(S \otimes \Lambda)^G$ is a graded K-vector space. We wish to compute its Poincaré series. In fact we will find two expressions for this series. In analogy

with our examination of the Poincaré series of S^G, one will be obtained from the structure of $(S \otimes \Lambda)^G$ given by the theorem and the other will result from the examination of the eigenvalues of the transformation.

Let a set of basic invariants of S^G be given by $f_1, \ldots, f_n$ of respective degrees $m_1 + 1 \leq \cdots \leq m_n + 1$

Proposition 7.1.4. *[Slm] The Poincaré series of the K-algebra of invariant differential p forms $(S \otimes \Lambda^p)^G$ is given by:*

(i) $\frac{\sigma_p(t^{m_1}, \cdots, t^{m_n})}{(1-t^{m_1+1}) \cdots (1-t^{m_n+1})}$ where σ_p denotes the pth symmetric polynomial in n elements.

(ii) $\frac{1}{|G|} \sum_{\gamma \in G} \frac{\sigma_p(\omega_1(\gamma), \cdots, \omega_n(\gamma))}{(1-\omega_1(\gamma)t) \cdots (1-\omega_n(\gamma)t)}$ where $\omega_1(\gamma), \cdots, \omega_n(\gamma)$ are the eigenvalues of the automorphism induced on V by $\gamma \in G$.

Proof: To prove (i) we use our knowledge of the Poincaré series of $R = S^G$ and Proposition 5.1.6. In our case, $(S \otimes \Lambda^p)^G$ is a free graded S^G-module generated by the homogeneous elements $df_{i_1} \wedge \cdots \wedge df_{i_p}$ of degrees $\sum_{k=1}^{p} m_{i_k}$. By Proposition 4.2.2(b), we find that

$$\begin{aligned} P((S \otimes \Lambda^p)^G; t) &= \Big(\sum_{i_1 < \cdots < i_p} t^{m_{i_1} + \cdots + m_{i_p}} \Big) P_R(t) \\ &= \sum_{i_1 < \cdots < i_p} (t^{m_{i_1}}) \cdots (t^{m_{i_p}}) P_R(t) \\ &= \frac{\sigma_p(t^{m_1}, \cdots, t^{m_n})}{(1 - t^{m_1+1}) \cdots (1 - t^{m_n+1})} \end{aligned}$$

To prove (ii), we extend the field K to its algebraic closure. The grading of $(S \otimes \Lambda^p)^G$ is given by $\oplus_{q=0}^{\infty} (S^q \otimes \Lambda^p)^G$. Let $\gamma \in G$ and let $\Lambda^p(\gamma), S^q(\gamma)$ denote the canonical extensions of the automorphisms induced by γ on V to Λ^p and S^q respectively. Then $\tau_q(\gamma) = \mathrm{tr}(S^q(\gamma))$ and $\delta_p(\gamma) = \mathrm{tr}(\Lambda^p(\gamma))$ are the respective characters of the representations of G in S^q and Λ^p where both τ_0 and δ_0 denote the trivial character. $\tau_q \delta_p$ is the character of the representation of G on $S^q \otimes \Lambda^p$. But then, Lemma 5.2.1 states that if G has a representation over a K-vector space,

the dimension of the space of invariants is the average over G of the character of the representation. Hence $\dim(S^q \otimes \Lambda^p) = \frac{1}{|G|}\sum_{\gamma\in G} \tau_q(\gamma)\delta_p(\gamma)$. Let $\omega_1(\gamma), \cdots, \omega_n(\gamma)$ be the eigenvalues of the linear tranformation $\gamma \in G$. Proposition 5.2.2 shows that

$$\sum_{q=0}^{\infty} \tau_q(\gamma)t^q = \frac{1}{(1-\omega_1(\gamma)t)\cdots(1-\omega_n(\gamma)t)}$$

$$\delta_p(\gamma) = \sigma_p(\omega_1(\gamma), \cdots, \omega_n(\gamma))$$

Hence combining all the above equations,

$$\begin{aligned} P(S\otimes\Lambda^p)^G;t) &= \sum_{q=0}^{\infty} \dim(S^q\otimes\Lambda^p)t^q \\ &= \frac{1}{|G|}\sum_{\gamma\in G} \frac{\sigma_p(\omega_1(\gamma),\cdots,\omega_n(\gamma))}{(1-\omega_1(\gamma)t)\cdots(1-\omega_n(\gamma)t)} \end{aligned}$$

Recall that in the proof of the formula calculating the number of reflections in $G.$, we equated two expressions for the Poincaré series of S^G. We will follow the same method here to prove Solomon's result and its generalization. As usual, a few preliminary lemmas are required.

Lemma 7.1.5. *Let $\alpha = (\alpha_1, \cdots, \alpha_n) \in K^n$. Then*

$$\sum_{i_1<\cdots<i_p} \frac{\alpha_{i_1}\cdots\alpha_{i_p}}{(1-\alpha_{i_1}t)\cdots(1-\alpha_{i_p}t))} = \sum_{q=0}^{n} \frac{h_{pq}(t)\delta_q(\alpha)}{(1-\alpha_1 t))\cdots(1-\alpha_n t)}$$

for some $h_{pq}(t) \in K[T]$ where $\delta_q(\alpha) = \sigma_q(\alpha_1, \cdots, \alpha_n)$.

Proof: This result looks deceptively complicated. In fact it is a consequence of taking the common denominator $(1-\alpha_1 t)\cdots(1-\alpha_n t)$ for all terms on the left hand side. Recall that by definition

$$\sum_{p=0}^{m}(-1)^p\sigma_p(\alpha_1,\ldots,\alpha_m)t^p = (1-\alpha_1 t)\cdots(1-\alpha_m t)$$

where $\sigma_p(\alpha_1,\ldots,\alpha_m)$ is the pth symmetric polynomial on the m elements $\alpha_1,\ldots,\alpha_m$ and that

$$\sigma_p(\alpha_1,\ldots,\alpha_m) = \sum_{i_1<\cdots<i_p} \alpha_{i_1}\cdots\alpha_{i_p}$$

contains $\binom{m}{p}$ terms. We actually prove for $I \subset \{1, \cdots, n\}$

$$\sum_{|I|=p} \prod_{i \in I} \alpha_i \prod_{j \notin I} (1 - \alpha_j t) = \sum_{i=1}^{n} h_{pi}(t) \delta_i(\alpha)$$

Then

$$\begin{aligned}
&\sum_{|I|=p} \prod_{i \in I} \alpha_i \prod_{j \notin I} (1 - \alpha_j t) \\
&\quad = \sum_{|I|=p} \prod_{i \in I} \alpha_i \sum_{q=0}^{n-p} (-1)^q \sigma_q((\alpha_j)_{j \notin I}) t^q \\
&\quad = \sum_{|I|=p} \prod_{i \in I} \alpha_i \sum_{k=p}^{n} (-1)^{k-p} \sigma_{k-p}((\alpha_j)_{j \notin I}) t^{k-p} \\
&\quad = \sum_{k=p}^{n} (-1)^{k-p} \binom{k}{p} \sigma_k(\alpha_1, \ldots, \alpha_n) t^{k-p} \\
&\quad = \sum_{k=p}^{n} (-1)^{k-p} \binom{k}{p} t^{k-p} \delta_k(\alpha)
\end{aligned}$$

To justify the second last step we look at the coefficient of each term $\alpha_{i_1} \cdots \alpha_{i_k}$. For $k < p$, this is 0. For $k \geq p$ the term is obtained from $\binom{k}{p}$ terms of the form

$$(-1)^{k-p} t^{k-p} \left[\prod_{i \in I} \alpha_i \right] \sigma_{k-p}((\alpha_j)_{j \notin I}).$$

Hence we have our result with

$$h_{pk}(t) = \begin{cases} (-1)^{k-p} \binom{k}{p} t^{k-p} & \text{if } k \geq p \\ 0 & \text{if } k < p \, . \end{cases}$$

∎

We will apply this lemma to the case $\alpha_i = \omega_i(\gamma)$ for $i = 1, \cdots, n$ where the $\omega_i(\gamma)$ are the eigenvalues of $\gamma \in G$ on V. The importance of this lemma lies in the fact that it shows that $\sum_{|I|=p} \prod_{i \in I} \frac{\omega_i(\gamma)}{1 - \omega_i(\gamma) t}$ is a linear combination of the characters of the representation of G in Λ^p with coefficients which are independent of the group element. This will allow us to prove the following lemma.

Lemma 7.1.6. *[Slm] Let the eigenvalues of the linear transformation induced by $\gamma \in G$ be $\omega_1(\gamma), \cdots, \omega_n(\gamma)$. Let the exponents of G be $m_1 \leq \cdots \leq m_n$. Then*

$$\frac{1}{|G|} \sum_{\gamma \in G} \sum_{|I|=p} \prod_{i \in I} \frac{\omega_i(\gamma)}{1-\omega_i(\gamma)t} = \sum_{|I|=p} \prod_{i \in I} \frac{t^{m_i}}{1-t^{m_i+1}}$$

for all $p = 1, \ldots, n$.

Proof: Equating the two expressions given for the Poincaré series of $(S \otimes \Lambda^p)^G$ in Proposition 7.1.4, we obtain

$$\frac{1}{|G|} \sum_{\gamma \in G} \frac{\sigma_p(\omega_1(\gamma), \ldots, \omega_n(\gamma))}{(1-\omega_1(\gamma)t)\cdots(1-\omega_n(\gamma)t)} = \frac{\sigma_p(t^{m_1}, \ldots, t^{m_n})}{(1-t^{m_1+1})\cdots(1-t^{m_n+1})}$$

for $p = 1, \cdots, n$ and even for $p = 0$ if the zeroth symmetric polynomial is set to be 1. In other words a function of the form $\frac{\delta_p(\gamma)}{(1-\omega_1(\gamma)t)\cdots(1-\omega_n(\gamma)t)}$ may be averaged over the group G by replacing $\omega_i(\gamma)$ by t^{m_i} wherever it occurs. Hence we may average the result of the last lemma for $\alpha_i = \omega_i(\gamma)$ over the group.

$$\begin{aligned}
\frac{1}{|G|} \sum_{\gamma \in G} \sum_{|I|=p} \prod_{i \in I} \frac{\omega_i(\gamma)}{1-\omega_i(\gamma)t} &= \frac{1}{|G|} \sum_{\gamma \in G} \sum_{q=1}^{n} \frac{h_{pq}(t)\delta_q(\gamma)}{(1-\omega_1(\gamma)t)\cdots(1-\omega_n(\gamma)t)} \\
&= \sum_{q=1}^{n} \frac{h_{pq}(t)\sigma_q(t^{m_1}, \cdots, t^{m_n})}{(1-t^{m_1+1})\cdots(1-t^{m_n+1})} \\
&= \sum_{|I|=p} \prod_{i \in I} \frac{t^{m_i}}{1-t^{m_i+1}}
\end{aligned}$$

Note that in the last step we reapply Lemma 7.1.5 with $\alpha_i = t^{m_i}$. ∎

In the case of a finite real reflection group W, Solomon's result was generalized by Weiss and Pianzola to determine the number of elements of W with $e^{\frac{2\pi i}{m}}$ as an eigenvalue of a given multiplicity. We will prove the generalization for W and then show how Solomon's result holds for a finite reflection group G over an arbitrary field K of characteristic 0.

Definition 7.1.7. For $\gamma \in G$, $e_m(\gamma)$ is the multiplicity of a given primitive mth root of unity as an eigenvalue of the linear transformation induced by γ on V. In

particular for a finite real reflection group W, $e_m(w)$ is the multiplicity of $e^{\frac{2\pi i}{m}}$ as an eigenvalue of w on V.

Lemma 7.1.8. *[PW] For a finite real reflection group W with exponents $m_1 \leq \cdots \leq m_n$ we have*

$$\frac{1}{|W|}\sum_{w\in W}\binom{e_m(w)}{q} = \sum_{I\subset I(m),|I|=q}\prod_{i\in I}\frac{1}{m_i+1}$$

for each subset I of $\{1,\cdots,n\}$ where $I(m)=\{1\leq i\leq n | m|m_i+1\}$ and $q\geq 0$.

Proof: From Lemma 7.1.6 , we see

$$\frac{1}{|W|}\sum_{w\in W}\sum_{|I|=q}\prod_{i\in I}\frac{1}{\omega_i(w)^{-1}-t} = \sum_{|I|=q}\prod_{i\in I}\frac{t^{m_i}}{1-t^{m_i+1}}$$

where $\{\omega_1(w),\cdots,\omega_n(w)\}$ is the set of eigenvalues of $w\in W$ and I varies over subsets of $\{1,\cdots,n\}$. We will equate the sum of residues from the poles of order q at $t=\xi$ from both sides of the equation.

On the left hand side only those terms corresponding to elements $w\in W$ with ξ^{-1} (and hence ξ) as an eigenvalue of multiplicity at least q can contribute. In this case $e_m(w)\geq q$ and each such $w\in W$ contributes $\sum_{|I|=q,I\subset\{i|\omega_i(w)=\xi^{-1}\}}(-1)^q = (-1)^q\binom{e_m(w)}{q}$. Summing over all such terms from the left hand side we obtain $\frac{1}{|W|}\sum_{w\in W}(-1)^q\binom{e_m(w)}{q}$.

On the right hand side, a pole of order q at $t=\xi$ occurs exactly when $(\xi-t)|(1-t^{m_i+1})$ for all $i\in I$ for a given I. But this occurs if and only if $m|m_i+1$ for all i if and only if $I\subset I(m)$. To determine the residue for such a subset I we examine the Laurent expansion at $t=\xi$. We obtain

$$\lim_{t\to\xi}\prod_{i\in I}\frac{t^{m_i}(t-\xi)}{1-t^{m_i+1}} = \lim_{t\to\xi}\prod_{i\in I}\frac{t^{m_i}(-1)}{(t^{m_i}+\xi t^{m_i-1}+\cdots+\xi^{m_i})}$$

$$=\prod_{i\in I}\frac{-1}{m_i+1} = (-1)^{|I|}\prod_{i\in I}\frac{1}{m_i+1}$$

So the right hand side contributes

$$\sum_{I\subset I(m),|I|=q}(-1)^q\prod_{i\in I}\frac{1}{m_i+1}$$

as required. ∎

Finally we may prove the theorem.

Theorem 7.1.9. *(Pianzola-Weiss)[PW] Let W be a finite real reflection group with exponents $m_1\leq\cdots\leq m_n$. Then*

$$\frac{1}{|W|}\sum_{k=0}^{n}s_k(m)t^k=\frac{1}{|W|}\sum_{w\in W}t^{e_m(w)}=\prod_{i\in I(m)}\frac{m_i+t}{m_i+1}$$

where $I(m)=\{1\leq i\leq n|m|m_i+1\}$, $s_k(m)$= the number of elements of W which admit a primitive m^{th} root of unity as an eigenvalue of multiplicity k and $e_m(w)=$ multiplicity of a primitive m^{th} root of unity as an eigenvalue of w.

Proof: Clearly the first equality holds. We show that the last two polynomials have the same expansions in powers of t-1. The coefficient of $(t-1)^q$ in $\frac{1}{|W|}\sum_{w\in W}t^{e_m(w)}$ is

$$\frac{1}{q!}\frac{d^q}{dt^q}\left(\frac{1}{|W|}\sum_{w\in W}t^{e_m(w)}\right)|_{t=1}=\frac{1}{|W|}\sum_{w\in W}\binom{e_m(w)}{q}$$

On the other hand, the coefficient of $(t-1)^q$ in the third polynomial is

$$\begin{aligned}&\frac{1}{q!}\frac{d^q}{dt^q}\left(\prod_{i\in I(m)}\frac{m_i+t}{m_i+1}\right)|_{t=1}\\&=\sum_{I\subset I(m),|I|=q}\prod_{i\in I}\frac{d}{dt}\left(\frac{m_i+t}{m_i+1}\right)\prod_{i\notin I}\frac{t+m_i}{1+m_i}|_{t=1}\\&=\sum_{I\subset I(m),|I|=q}\prod_{i\in I}\frac{1}{m_i+1}\end{aligned}$$

But by Lemma 7.1.8, the coefficients of $(t-1)^q$ in the two polynomials are the same. This concludes the proof. ∎

Theorem 7.1.10. *(Solomon)[Slm] Let G be a finite reflection subgroup of $GL(V)$ where V is a vector space over a field K of characteristic 0. Let the exponents of G be $m_1 \leq \cdots \leq m_n$. Then*

$$\sum_{k=0}^{n} s_k(1)t^k = \sum_{\gamma \in G} t^{c_1(\gamma)} = \prod_{i=1}^{n}(t + m_i)$$

Proof: Note that the proof of Theorem 7.1.9 hinges on Lemma 7.1.8. But the proof of Lemma 7.1.8 still holds for G in the case m=1. So since the result of Theorem 7.1.9 for m=1 is valid for G and $|G| = \prod_{i=1}^{n}(m_i + 1)$, the theorem follows. ∎

CHAPTER 8

Solomon with Character

In this section, we examine the formula $\sum_{w \in W} \chi(w) t^{e_1(w)}$ for a character χ of the finite reflection subgroup W of $GL(V)$ where V is a real euclidean n-space. We wish to show that at least in particular cases this polynomial factors into integral linear factors involving the exponents of W. This will generalize Solomon's result which gives such a factorization for the trivial character. We are particularly interested in the formula for irreducible characters. If this could be determined for all irreducible characters, Theorem 1.2.6 tells us that the formula for any arbitrary character χ could be determined given a knowledge of the multiplicities of the irreducible characters in χ.

By 1.2.6, and the fact that $\epsilon(w) = \det w = \pm 1$ for all $w \in W$ we find that the determinant character ϵ is irreducible. In fact $\epsilon\chi$ is irreducible for an arbitrary irreducible character χ since $(\epsilon\chi, \epsilon\chi) = (\chi, \chi) = 1$.

Proposition 8.1.1. *Let W be a finite real reflection group with exponents $m_1 \leq \cdots \leq m_n$. Then*

(i) $\sum_{w \in W} \epsilon(w)\chi(w) t^{e_1(w)} = (-1)^n \sum_{w \in W} \chi(w)(-t)^{e_1(w)}$ *for an arbitrary character χ of W.*

(ii) $\sum_{w \in W} \epsilon(w) t^{e_1(w)} = \prod_{i=1}^{n} (t - m_i)$.

(iii) $\frac{1}{|W|} \sum_{w \in W} \epsilon(w) t^{e_2(w)} = \prod_{i \in I(2)} \frac{(m_i - t)}{(m_i + 1)}$

Proof: We show that $\epsilon(w) = (-1)^{n - e_1(w)}$. The eigenvalues of w are mth roots of unity where m is the order of w. Since w is a real linear transformation, the complex eigenvalues occur in r complex conjugate pairs. Then $n = e_1(w) + e_2(w) + 2r$. Since the determinant of w is the product of the eigenvalues, we have $\det w = (-1)^{n - e_1(w) - 2r} = (-1)^{n - e_1(w)}$. Then (i) follows immediately. (ii) is proven using

(i) with $\chi = 1_W$ and Solomon's result:

$$\sum_{w \in W} \epsilon(w) t^{e_1(w)} = (-1)^n \prod_{i=1}^{n} ((-t) + m_i) = \prod_{i=1}^{n} (t - m_i).$$

(iii) follows from the generalization of Solomon's result in 7.1.9 and the fact that $\epsilon(w) = (-1)^{e_2(w)}$. ∎

We next want to examine the formula for the character $1_{W_J}^W$ corresponding to the coset representation of a parabolic subgroup W_J in W. Although these characters are generally not irreducible, at least in some cases the knowledge of this factorization can be used to obtain the factorization of $\sum_{w \in W} \chi(w) t^{e_1(w)}$ for an irreducible character χ of W.

We first require a technical lemma.

Lemma 8.1.2. *Let G be a finite subgroup of $GL(V)$ where $G \cong \prod_{i=0}^{k} G_i$ and $V = \oplus_{i=0}^{k} V_i$ are such that:*

(i) Each V_i is G-stable.

(ii) $g_i |_{V_j} = Id |_{V_j}$ where $g_i \in G_i, i \neq j$. Then $e_1^V(g_0 \cdots g_k) = \sum_{i=0}^{k} e_1^{V_i}(g_i)$ where for all G-stable subspaces U of V we have $e_1^U(h) =$ multiplicity of 1 as an eigenvalue of h on U and $g_i \in G_i, i = 0, \cdots, k$.

Proof: We prove the result by induction on k. If k=0, the result is clear.

Assume the result holds for $i < k$. Set $g = g_0 \cdots g_{k-1}$, $U_k = \oplus_{i \neq k} V_i$. We first show

$$e_1^V(gg_k) = e_1^{V_k}(gg_k) + e_1^{U_k}(gg_k) \qquad (*)$$

Clearly

$$e_1^V(gg_k) \geq e_1^{V_k}(gg_k) + e_1^{U_k}(gg_k)$$

since a vector in V_k or U_k fixed by gg_k is also fixed by gg_k in V.

Suppose $gg_k(v_k + u_k) = v_k + u_k$ where $v_k \in V_k, u_k \in U_k$. Then $gg_k(v_k) = x_k \in V_k$ and $gg_k(u_k) = y_k \in U_k$ since V_k and U_k are G-stable. But then $x_k + y_k = v_k + u_k$

and the uniqueness of the decomposition $V = V_k \oplus U_k$ imply $gg_k(v_k) = v_k$ and $gg_k(u_k) = u_k$. Hence (*) follows.

But by (i) and (ii), $gg_k \mid_{V_k} = g_k \mid_{V_k}$ and $gg_k \mid_{U_k} = g \mid_{U_k}$. We find $e_1^V(gg_k) = e_1^{V_k}(g_k) + e_1^{U_k}(g)$. Hence the result follows from the inductive hypothesis. ∎

Corollary 8.1.3. *Let W be a finite real reflection group of V with root system Φ with base Δ.*

(i) Let $J \subset \Delta$. A parabolic subgroup W_J of W with Coxeter graph Γ_J having connected components $\Gamma_{J_1}, \cdots, \Gamma_{J_k}$ satisfies the hypotheses of the lemma where

$$V_0 = V^{W_J} = V_{\Delta - J} \qquad V_i = V_{J_i}, i = 1, \cdots k$$

$$W_0 = \{1\} \qquad W_i = W_{J_i}, i = 1, \cdots, k$$

(ii) $e_1^V(w_1 \cdots w_k) = \dim V^{W_J} + \sum_{i=1}^k e_1^{V_i}(w_i)$ where $w_i \in W_i, i = 1, \cdots, k$.

Proof: The first statement follows from Proposition 3.4.6. The second statement results from the lemma and the observation that $e_1^{V^{W_J}}(1) = \dim V^{W_J}$. ∎

Theorem 8.1.4. *With the notation of Corollary 8.1.3 we have*

$$\sum_{w \in W} 1_{W_J}^W(w) t^{e_1^V(w)} = \frac{|W|}{|W_J|} \sum_{w \in W_J} t^{e_1^V(w)}$$

$$= \frac{|W|}{|W_J|} t^{\dim V^{W_J}} \left[\prod_{i=1}^k \Big(\sum_{w_i \in W_i} t^{e_1^{V_i}(w_i)} \Big) \right]$$

Proof: Let $p = \frac{|W|}{|W_J|}$ and $W = \cup_{i=1}^p x_i W_J$ be the coset decomposition of W with respect to W_J. Then

$$1_{W_J}^W(w) = |\{i | w x_i W_J = x_i W_J\}|$$

$$= |\{i | x_i^{-1} w x_i \in W_J\}|$$

$$= \sum_{i=1}^p 1_{W_J}(x_i^{-1} w x_i)$$

where

$$1_{W_J}(y) = \begin{cases} 1 & \text{if } y \in W_J \\ 0 & \text{if } y \notin W_J. \end{cases}$$

$$\begin{aligned}
\sum_{w\in W} 1_{W_J}^{W}(w)t^{e_1^V(w)} &= \sum_{w\in W}(\sum_{i=1}^{p} 1_{W_J}(x_i^{-1}wx_i))t^{e_1^V(w)} \\
&= \sum_{i=1}^{p}\sum_{w\in W} 1_{W_J}(x_i^{-1}wx_i)t^{e_1^V(w)} \\
&= \sum_{i=1}^{p}\sum_{w\in x_i^{-1}W_Jx_i} t^{e_1^V(w)} \\
&= \sum_{i=1}^{p}\sum_{y\in W_J} t^{e_1^V(x_iyx_i^{-1})} \\
&= p\sum_{y\in W_J} t^{e_1^V(y)} \\
&= \frac{|W|}{|W_J|}\sum_{y_1\cdots y_k\in W_1\cdots W_k} t^{dimV^{W_J}+e_1^{V_1}(y_1)+\cdots+e_1^{V_k}(y_k)} \\
&= \frac{|W|}{|W_J|}t^{dimV^{W_J}}\prod_{i=1}^{k}\sum_{y_i\in W_i} t^{e_1^{V_i}(y_i)}
\end{aligned}$$

∎

Hence using the theorem, we may "paste" together the factorizations for the trivial character of the irreducible real reflection groups W_i to find the factorization for the character $1_{W_J}^{W}$.

The theorem produces an interesting result in the case $W = S_n$ acting naturally on $\mathbf{R}^n = \oplus_{i=1}^{n}\mathbf{R}\epsilon_i$. Here a parabolic subgroup is isomorphic to a Young subgroup $S_\lambda = S_{\lambda_1}\times\cdots\times S_{\lambda_k}$ corresponding to a partition $\lambda = (\lambda_1,\cdots,\lambda_k)$ of n.

Corollary 8.1.5. *Let $W = S_n$ and $\lambda = (\lambda_1,\cdots,\lambda_k)\mapsto n$. Then*

$$\begin{aligned}
\sum_{w\in S_n} 1_{S_\lambda}^{S_n}(w)t^{e_1^{\mathbf{R}^n}}(w) &= \frac{|S_n|}{|S_\lambda|}\prod_{i=1}^{k}[t(t+1)\cdots(t+\lambda_i-1)] \\
&= \frac{|S_n|}{|S_\lambda|}\prod_{i=1}^{m}(t+i-1)^{\mu_i}
\end{aligned}$$

where $\mu = (\mu_1,\cdots,\mu_m)\mapsto n$ is the dual partition of λ. For $V = (\sum_{i=1}^{n}\epsilon_i)^\perp$,

$$\sum_{w\in S_n} 1_{S_\lambda}^{S_n}(w)t^{e_1^V}(w) = \frac{1}{t}\sum_{w\in S_n} 1_{S_\lambda}^{S_n}(w)t^{e_1^{\mathbf{R}^n}}(w)$$

Proof: Here we have a decomposition $\mathbf{R}^n \cong \oplus_{i=1}^k \mathbf{R}^{\lambda_i}$ corresponding to $S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_k}$ satisfying the hypotheses of Lemma 8.1.2. Note that the exponents of S_{λ_i} acting on $\mathbf{R}^{\lambda_i}$ are $0, 1, \cdots, \lambda_i - 1$ so that $\sum_{w_i \in S_{\lambda_i}} t^{e_1^{\mathbf{R}^{\lambda_i}}(w_i)} = \prod_{j=1}^{\lambda_i}(t+j-1)$. From the lemma and the first equality of the theorem we obtain:

$$\sum_{w \in S_n} 1_{S_\lambda}^{S_n}(w) t^{e_1^{\mathbf{R}^n}}(w) = \frac{|S_n|}{|S_\lambda|} \prod_{i=1}^{k} \sum_{w_i \in W_i} t^{e_1^{V_i}}(w_i)$$

$$= \frac{|S_n|}{|S_\lambda|} \prod_{i=1}^{k} \prod_{j=1}^{\lambda_i} (t+j-1) \qquad (*)$$

The second equality results from rearranging the factors in (*). We find (*) $= \frac{|S_n|}{|S_\lambda|} \prod_{j=1}^{m} (t+j-1)^{\mu_j}$ where $m = \lambda_1 - 1$ and

$$\mu_j = \#\text{ of i such that } j \geq \lambda_i$$

But then $\mu = (\mu_1, \cdots, \mu_m)$ is precisely the dual partition of λ.

For the last statement, note that the decomposition $\mathbf{R}^n = V^\perp \oplus V$ corresponding to $\{1\}$ and S_n satisfy the hypotheses of the lemma. Thus

$$\sum_{w \in S_n} 1_{S_\lambda}^{S_n}(w) t^{e_1^{\mathbf{R}^n}(w)} = t \sum_{w \in S_n} 1_{S_\lambda}^{S_n}(w) t^{e_1^{V}(w)}$$

The result follows. ∎

We can now determine the formula for the natural character of $\mathcal{A}_n$.

Corollary 8.1.6.

$$\sum_{w \in \mathcal{A}_n} \chi^{(n,1)}(w) t^{e_1(w)} = n(t-1)\cdots(t+n-1)$$

where $S_{n+1} = \mathcal{A}_n$ acts on $V = (\mathbf{R}(\sum_{i=1}^{n+1} \epsilon_i))^\perp$.

Proof: By 1.3.5, Corollary 8.1.5 and Solomon's result we find

$$\sum_{w \in \mathcal{A}_n} \chi^{(n,1)} t^{e_1(w)} = \sum_{w \in \mathcal{A}_n} (1_{S_{(n,1)}}^{S_{n+1}} - 1_{S_{n+1}}) t^{e_1(w)}$$

$$= (n+1)t(t+1)\cdots(t+n-1) - (t+1)\cdots(t+n) \quad ∎$$

$$= n(t-1)(t+1)\cdots(t+n-1)$$

Example 8.1.7. Let $\lambda = (n^{k_n}, \cdots, 1^{k_1})$ be the cycle type of elements in the conjugacy class $C(n,\lambda)$ where $w \in C(n,\lambda)$ has k_i cycles of length i. Recall that $|C(n,\lambda)| = \frac{n!}{n^{k_n}k_n!\cdots 1^{k_1}k_1!}$. For notational convenience we omit i in λ if $k_i = 0$. Then for $w \in C(n,\lambda)$ we have

$$\chi(n,\lambda) := \chi(w) = k_1 - 1$$

$$e_1(n,\lambda) := e_1(w) = \text{ number of cycles in } w - 1$$

For $\mathcal{A}_4 = S_5$ we have

Table 8.1.1: Data for $\mathcal{A}_4$

$C(n,\lambda)$	$\lvert C(n,\lambda)\rvert$	$e_1(n,\lambda)$	$\chi(n,\lambda)$	$\epsilon(n,\lambda)$
(1^5)	1	4	4	1
$(2,1^3)$	10	3	2	-1
$(3,1^2)$	20	2	1	1
$(2^2,1)$	15	2	0	1
$(4,1)$	30	1	0	-1
$(3,2)$	20	1	-1	-1
(5)	24	0	-1	1

Then

$$\sum_{w\in\mathcal{A}_n} t^{e_1(w)} = t^4 + 10t^3 + 35t^2 + 50t + 24 = (t+1)(t+2)(t+3)(t+4)$$

$$\sum_{w\in\mathcal{A}_n} \epsilon(w)t^{e_1(w)} = t^4 - 10t^3 + 35t^2 - 50t + 24 = (t-1)(t-2)(t-3)(t-4)$$

$$\sum_{w\in\mathcal{A}_n} \chi(w)t^{e_1(w)} = 4t^4 + 20t^3 + 20t^2 - 20t - 24 = 4(t-1)(t+1)(t+2)(t+3)$$

In general we propose the following (unproven) conjecture:

Conjecture 1. *Let χ be the natural character of a finite real reflection subgroup W of V with exponents $m_1 \le \cdots \le m_n$. Then*

$$\sum_{w\in W} \chi(w)t^{e_1(w)} = n(t-1)\prod_{i=1}^{n-1}(t+m_i).$$

We see that Corollary 8.1.6 agrees with Conjecture 1 for $W = \mathcal{A}_n$. We have also verified this conjecture computationally for dimensions $n \leq 4$ for $\mathcal{B}_n$, $\mathcal{D}_n$ and for $\mathcal{F}_4$ and $\mathcal{G}_2$.

Example 8.1.8. $W = \mathcal{B}_n$

A conjugacy class $K(n,\lambda)$ of $\mathcal{B}_n$ may be represented by $\lambda = (n^{(p_n,d_n)}, \cdots, 1^{(p_1,d_1)})$ where $w \in K(n,\lambda)$ has p_i positive cycles of length i and d_i negative cycles of length i. Again i is omitted in λ if $p_i = d_i = 0$. Then

$$|K(n,\lambda)| = \prod_{i=1}^{n} \binom{p_i + d_i}{p_i} \prod_{j \geq 2, p_j + d_j > 0} 2^{j-1} |C(n,\lambda)|$$

$$\chi(n,\lambda) = p_1 - n_1$$

$$e_1(n,\lambda) = \sum_{i=1}^{n} p_i$$

For $\mathcal{B}_3$ we have

Table 8.1.2: Data for $\mathcal{B}_3$

$K(n,\lambda)$	$\lvert K(n,\lambda)\rvert$	$e_1(n,\lambda)$	$\chi(n,\lambda)$	$\epsilon(n,\lambda)$
$(1^{(3,0)})$	1	3	3	1
$(1^{(2,1)})$	3	2	1	−1
$(1^{(1,2)})$	3	1	−1	1
$(1^{(0,3)})$	1	0	−3	−1
$(2^{(1,0)}, 1^{(1,0)})$	6	2	1	−1
$(2^{(1,0)}, 1^{(0,1)})$	6	1	−1	1
$(2^{(0,1)}, 1^{(1,0)})$	6	1	1	1
$(2^{(0,1)}, 1^{(0,1)})$	6	0	−1	−1
$(3^{(1,0)})$	8	1	0	1
$(3^{(0,1)})$	8	0	0	−1

Then

$$\sum_{w \in B_3} t^{e_1(w)} = t^3 + 9t^2 + 23t + 15 = (t+1)(t+3)(t+5)$$

$$\sum_{w \in B_3} \chi(w) t^{e_1(w)} = 3t^3 + 9t^2 - 3t - 9 = 3(t-1)(t+1)(t+3)$$

$$\sum_{w \in B_3} \epsilon(w) t^{e_1(w)} = t^3 - 9t^2 + 23t - 15 = (t-1)(t-3)(t-5)$$

We may also generalize the Corollary in another way. The formula $1_{S_\mu}^{S_n} = \sum_{\lambda \geq \mu} K_{\lambda\mu} \chi^\lambda$ can be used recursively to find the formula for the irreducible character χ^λ of S_n corresponding to a partition λ of n. Although this can be done on a case by case basis, I have not yet found an efficient way to compute a general formula for a given χ^λ. However, from an examination of the low dimensional cases we have an interesting conjecture:

Conjecture 2. *Let S_n act naturally on $\mathbf{R}^n$. Then*

$$\sum_{w \in S_n} \chi^\lambda(w) t^{e_1(w)} = \chi^\lambda(1) \prod_{(i,j) \in \lambda} (t + j - i)$$

where λ is a partition of n,

$$\chi^\lambda(1) = deg\chi^\lambda = \text{ \# of standard tableaux of shape } \lambda,$$

$$(i,j) \in \lambda \Leftrightarrow \text{ The Young diagram of shape } \lambda \text{ has a box in row } i \text{ and column } j$$

Note that Corollary 8.1.6 with the adjustment factor given by 8.1.5 is a special case of this conjecture. For certain other special cases we can obtain the factorization.

Example 8.1.9. For S_n acting on $\mathbf{R}^n$ we have

$$\sum_{w \in S_n} \chi^{(n-k,k)}(w) t^{e_1(w)} = \binom{n}{k} \frac{n-2k+1}{n-k+1} (t-1) \prod_{i=0}^{k-2} (t+i)^2 \prod_{j=k-1}^{n-k-1} (t+j)$$

where $n - k \geq k$.

Let $\mu = (n-k,k)$. Claim that

$$1_{S_{(n-k,k)}}^{S_n} = \chi^{(n-k,k)} + 1_{S_{(n-k+1,k-1)}}^{S_n}.$$

Let $\lambda \mapsto n$ be such that $\lambda \geq \mu$. Then $\lambda_1 \geq n-k$ and $\lambda_1 + \lambda_2 \geq n$ implies that $\lambda = (n-k+r, k-r)$ with $0 \leq r \leq k$. Note that $K_{\lambda\mu} = 1$ for each such λ. Then

$$\begin{aligned} 1_{S_{(n-k,k)}}^{S_n} &= \sum_{r=0}^{k} \chi^{(n-k+r,k-r)} \\ &= 1_{S_{(n-k+1,k-1)}} + \chi^{(n-k,k)} \end{aligned}$$

So

$$\begin{aligned} &\sum_{w \in S_n} \chi^{(n-k,k)}(w) t^{e_1(w)} \\ &= \binom{n}{k} t^2(t+1)^2 \cdots (t+k-1)^2 (t+k) \cdots (t+n-k-1) \\ &- \binom{n}{k-1} t^2(t+1)^2 \cdots (t+k-2)^2 (t+k-1) \cdots (t+n-k) \\ &= \binom{n}{k} \frac{(n-2k+1)}{(n-k+1)} (t-1) t^2 (t+1)^2 \cdots (t+k-2)^2 (t+k-1) \cdots (t+n-k-1) \end{aligned}$$

Example 8.1.10. Let us determine the formulae for all the irreducible characters of S_5. We have already computed $\chi^{(5)}, \chi^{(4,1)}, \chi^{(1^5)} = \epsilon$. The result for $\chi^{(3,2)}$ follows from the previous example:

$$\sum_{w \in S_5} \chi^{(3,2)}(w) t^{e_1(w)} = 5(t-1)t^2(t+1)(t+2)$$

We need only find the formulae for $\chi^{(3,1^2)}$, $\chi^{(2^2,1)}$ and $\chi^{(2,1^3)}$. They follow from

$$\begin{aligned} 1_{S_{(3,1^2)}}^{S_5} &= \chi^{(3,1^2)} + \chi^{(3,2)} + 2\chi^{(4,1)} + \chi^{(5)} \\ 1_{S_{(2^2,1)}}^{S_5} &= \chi^{(2^2,1)} + \chi^{(3,1^2)} + 2\chi^{(3,2)} + 2\chi^{(4,1)} + \chi^{(5)} \\ 1_{S_{(2,1^3)}}^{S_5} &= \chi^{(2,1^3)} + 2\chi^{(2^2,1)} + 3\chi^{(3,1^2)} + 3\chi^{(3,2)} + 3\chi^{(4,1)} + \chi^{(5)} \end{aligned}$$

As predicted by the conjecture, we find that

$$\begin{aligned} \sum_{w \in S_5} \chi^{(3,1^2)}(w) t^{e_1(w)} &= 6(t-2)(t-1)t(t+1)(t+2) \\ \sum_{w \in S_5} \chi^{(2^2,1)}(w) t^{e_1(w)} &= 5(t-2)(t-1)t^2(t+1) \\ \sum_{w \in S_5} \chi^{(2,1^3)}(w) t^{e_1(w)} &= 4(t-3)(t-2)(t-1)t(t+1) \end{aligned}$$

These formulae have also been verified directly using the character table for S_3.

I intend to pursue this line of study further. I would like to prove the two conjectures and more generally examine the factorization of the polynomial

$$\sum_{w \in W} \chi(w) t^{e \cdot m(w)}$$

for an irreducible character χ of a finite real reflection group W.

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